This chapter is a brief introduction to the topic of vibrations. We already know about the harmonic oscillator, and that is core idea. But here we want to take account of three main ideas, with one section per idea.

1. **Damping.**There is friction, so oscillatory motion decays in time. This is simple enough conceptually, but the mathematical descriptions are more complicated than for the harmonic oscillator, involving a mix of exponential and sine functions.
2. **Forcing.** Things get pushed around, or ‘forced’. What is the relation between how much you push on something and how much it shakes. The key ideas here are resonance and frequency response.
3. **Normal modes.** Most real machines and structures have various moving parts. How do things move with various parts? For simplicity here we stick to the cases with no friction. The key idea is that one can think of the general motions as made up of (as a superposition of) simple harmonic motions. These are called normal modes.

10.1 Damped vibrations

In the real world, macroscopic oscillators that are not pumped have motions that decay in time due to dissipation. Our goal is to understand what happens to a harmonic oscillator when we add friction. In particular we are interested in the simplest kind of friction caused by an ideal linear dashpot. What we will find is exponentially decaying solutions that may, or may not have, an oscillatory nature depending on the amount of friction.

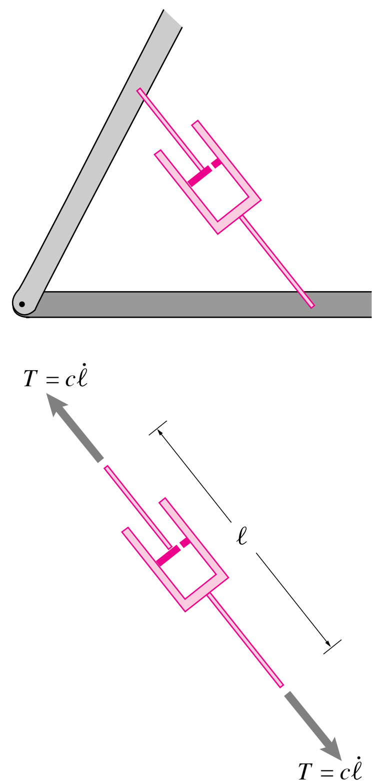

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Figure 10.2: **A dashpot.** A dashpot (or damper) is shown here connecting two parts of a mechanism. The tension in the dashpot is proportional to the rate at which it lengthens. The symbol shown represents any device which resists the relative motion of its endpoints. The schematic is supposed to suggest a plunger in a cylinder. For the plunger to move, fluid must leak around the cylinder. This leakage happens for either direction of motion. Thus the damper resists relative motion in either direction; i. e., for $\dot{\ell} > 0$ and $\dot{\ell} < 0$.

Damping

Dashpots are used to absorb energy. One is shown schematically in fig. 10.1. Often springs and dashpots are light in comparison to the machinery to which they are attached so their mass and weight are neglected. They are usually attached with pin joints, ball and socket joints, or other kinds of flexible connections so only forces are transmitted. Because they only have forces at their ends they are ‘two-force’ bodies so (see section 4.2) the forces at their ends are equal, opposite, and along the line of connection. The most familiar examples are the shock absorbers of a car or the damper for screen doors. The symbol for a dashpot shown in fig. 10.1.

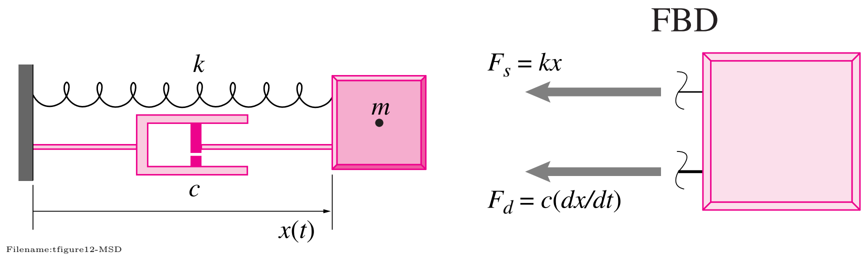

Figure 10.1: A mass spring dashpot system, or damped harmonic oscillator. Also shown is a free body diagram of the mass.

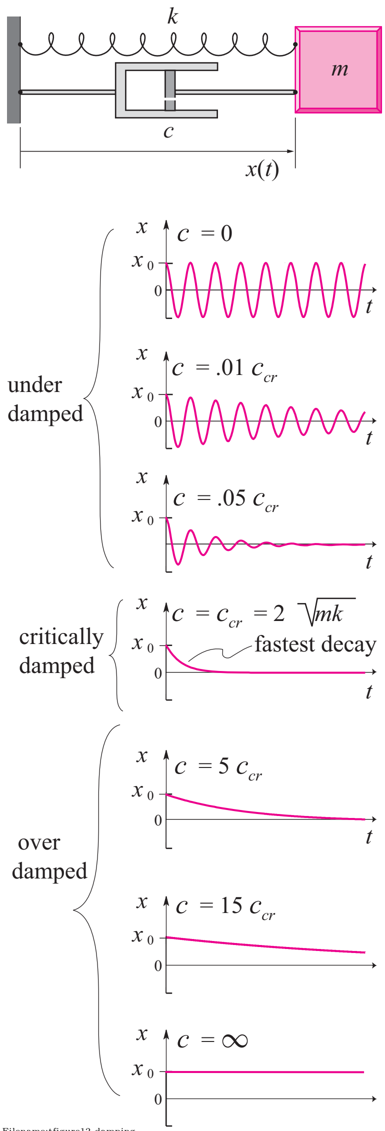

Figure 10.2: The effect of varying the damping with a fixed mass and spring. In all the plots the mass is released from rest at $x = x_0$. In the case of under-damping, oscillations persist for a long time, forever if there is no damping. In the case of over-damping, the dashpot doesn't relax for a long time; it stays locked up forever in the limit of $c \to \infty$. The fastest relaxation occurs for critical damping.

The dashpot provides resistance to motion by drawing air or oil in and out of the cylinder through a small opening. Due to the viscosity of the air or oil, a pressure drop is created across the opening that is related to the speed of the fluid flowing through. Ideally, this viscous resistance produces linear damping, meaning that the force is exactly proportional to the velocity. The relation is assumed to hold for negative lengthening as well. So the compression (negative tension) is also proportional to the rate at which the dashpot shortens (negative lengthens).

The tension in the dashpot is usually assumed to be proportional to the rate at which it lengthens, although this approximation is not especially accurate for most dampers one can buy. In a physical dashpot nonlinearities, from the fluid flow and from friction between the piston and the cylinder, are often significant. Also, dashpots that use air as a working fluid may have compressibility that introduces extra springiness to the system.

The defining equation for an ideal linear dashpot is:

$$T = C\,\dot{\ell}$$

where C is the dashpot constant.

Damped oscillations

We now add a dashpot in parallel with the spring of a mass-spring system creates a *mass-spring-dashpot* system, or *damped harmonic oscillator*. The system is shown in fig. 10.2. Also in fig. 10.2 is a free body diagram of the mass. It has two forces acting on it, neglecting gravity:

$$F_s = kx \quad \text{is the spring force, assuming a linear spring, and}$$
$$F_d = c\,dx/dt = c\dot{x} \quad \text{is the dashpot force assuming a linear dashpot.}$$

The system is a one degree of freedom system because a single coordinate x is sufficient to describe the complete motion of the system. The equation of motion for this system is

$$m\ddot{x} = -F_d - F_s \quad \text{where} \quad \ddot{x} = d^2x/dt^2. \tag{10.1}$$

Assuming a linear spring and a linear dashpot this expression becomes

$$m\ddot{x} + c\dot{x} + kx = 0. \tag{10.2}$$

We have taken care with the signs of the various terms. Make sure you can confidently derive equation 10.2 without introducing sign errors. The analytical solution of the damped-oscillator equation is in box 10.2. Some qualitative features of the damped solutions are shown in fig. 10.3

For given k and m we can think of the damping c as adjustable. A system which has small damping (small c) is *under-damped* and does not come to equilibrium quickly because oscillations last for a long time. A system which has a lot of damping (big c) is *over-damped* does not come to equilibrium quickly because the dashpot doesn't leak fast enough. A system which is in between, *critically-damped* comes to equilibrium most quickly. The purpose of damping is often to purge motion after a disturbance. If the only design variable available for adjustment is the damping, then the quickest purge is accomplished with critical damping, $c = \sqrt{(4km)}$. In practice, any damping value close to critical is often used, more or less depending on whether a little oscillation is tolerable or not①.

①Stereotypically, the suspension of an overloaded old-fashioned luxury car is underdamped, imagine it bouncing along after a bump. And the suspension of a tight sports car is underdamped.

Summary of equations for the unforced harmonic oscillator

- $\ddot{x} + \frac{k}{m}x = 0$, mass-spring equation
- $\ddot{x} + \lambda^2 x = 0$, harmonic oscillator equation
- $x(t) = A\cos(\lambda t) + B\sin(\lambda t)$, general solution to harmonic oscillator equation
- $x(t) = R\cos(\lambda t - \phi)$, amplitude-phase version of solution to harmonic oscillator solution, $R = \sqrt{A^2 + B^2}, \phi = \tan^{-1}(\frac{B}{A})$ (See box on page 460).
- $\ddot{x} + \frac{c}{m}\dot{x} + \frac{k}{m}x = 0$, mass-spring-dashpot equation (see equations 10.3-10.6 for solutions)

Solution of the damped-oscillator equations

Here are some mathematical details you can use for reference. These details are of much lower status than those in box C.1 on page 1016. Only some vibrations experts remember the formulas below in detail.

Even if we make the common assumptions that m, c, and k are all positive, the whole nature of the solution of (10.2) depends on the values of those constants. The three types of solutions are categorized as follows:

- *Under-damped*: $c^2 < 4mk$. In this case the damping is small and oscillations persist forever, though their amplitude diminishes exponentially in time. The general solution for this case is:

$$x(t) = e^{(-\frac{c}{2m})t}[A\cos(\lambda_d t) + B\sin(\lambda_d t)], \tag{10.3}$$

where λ_d is the damped natural frequency and is given by

$$\lambda_d = \sqrt{\frac{k}{m} - \left(\frac{c}{2m}\right)^2}. \tag{10.4}$$

- *Critically damped*: $c^2 = 4mk$. In this case the damping is at a critical level that separates the cases of under-damped oscillations from the simply decaying motion of the over-damped case. The general solution is:

$$x(t) = Ae^{(-\frac{c}{2m})t} + Bte^{(-\frac{c}{2m})t}. \tag{10.5}$$

- *Over-damped*: $c^2 > 4mk$. Here there are no oscillations, just a simple return to equilibrium with at most one crossing through the equilibrium position on the way to equilibrium. The general solution in the over-damped case is:

$$x(t) = Ae^{\left(-\frac{c}{2m}+\sqrt{(\frac{c}{2m})^2-\frac{k}{m}}\right)t} + Be^{\left(-\frac{c}{2m}-\sqrt{(\frac{c}{2m})^2-\frac{k}{m}}\right)t}. \tag{10.6}$$

Measurement of damping. In the under-damped case, the damping constant c can be found by measuring the rate of decay of unforced oscillations using the 'logarithmic decrement'. The logarithmic decrement is the natural logarithm of the ratio of the amplitude of any two successive peaks. The larger the damping, the greater the rate of decay and the bigger the decrement:

$$\text{logarithmic decrement} \equiv D = \ln(\frac{x_n}{x_{n+1}}) \tag{10.7}$$

where x_n and x_{n+1} are the heights of two successive peaks in the figure below (also seen on the 2nd and 3rd figures in fig. 10.3 on page 500). Because of the exponential envelope (bounding curve), $x_n = (\text{const.})e^{-(\frac{c}{2m})t_n}$ and $x_{n+1} = (\text{const.})e^{-(\frac{c}{2m})t_n+T}$.

$$D = \ln[(e^{-(\frac{c}{2m})t_n})/(e^{-(\frac{c}{2m})t_n+T})]$$

Simplifying this expression, we get that

$$D = \frac{cT}{2m}$$

where T is the period of oscillation. Thus, measuring the logarithmic decrement D and the period of oscillation T determines c as

$$c = \frac{2mD}{T}.$$

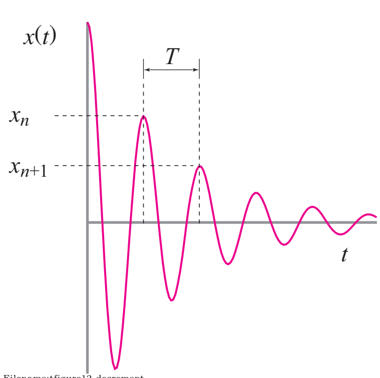

Figure 10.3: Measuring damping using 'logarithmic decrements'.

Solution using complex variables. To find the solutions above it is easiest to use complex variables. Consider this general constant coefficient 2nd order linear equation:

$$A\ddot{x} + B\dot{x} + Cx = 0. \tag{10.8}$$

We want to find the most general function $x(t)$ that satisfies this equation. We do this by making the *guess* that

$$x(t) = e^{\alpha t}.$$

Plugging this guess into eqn. (10.8) and cancelling the common non-zero factor $e^{\alpha t}$ from each term gives

$$A\alpha^2 + B\alpha + C = 0.$$

For simplicity let's assume this quadratic has two independent roots α_1 and α_2,

$$\alpha_{1,2} = \frac{-B \pm \sqrt{B^2 - 4AC}}{2A}.$$

For each root we have a solution to eqn. (10.8). And the solutions can be multiplied by a constant. And the solutions can be added. Thus the general solution to eqn. (10.8) is:

$$x(t) = C_1 e^{\alpha_1 t} + C_2 e^{\alpha_2 t}. \tag{10.9}$$

This seems easy enough. But the α's might be complex. And the C's too. To get a real solution we have to take the real part using the Euler equation $e^{i\lambda t} = \cos i\lambda t + \sin i\lambda t$). By this means, with lots of algebra, we could see that the solution 10.6 actually includes the solutions 10.3 and 10.5 as special cases. This algebra is carried out for the simplest case, no damping, in box 9.3 on page 466.

Numerical solution to the damped oscillator equations

As for the numerical solution of the harmonic oscillator we define $v = \dot{x}$. Thus

$$m\ddot{x} + c\dot{x} + kx = 0 \quad \Rightarrow \quad m\dot{v} + v + x = 0.$$

Combining the definition of v with the differential equation we get the set of two coupled first order equations

$$\begin{aligned} \dot{x} &= v \\ \dot{v} &= -\frac{k}{m}x - \frac{c}{m}v \end{aligned} \tag{10.10}$$

We can think of this as

$$\dot{z} = f(z)$$

where z is the list of two numbers $z(1) = x$ and $z(2) = v$ so

$$\frac{d}{dt}\begin{bmatrix} z_1 \\ z_2 \end{bmatrix} = \begin{bmatrix} z_2 \\ -\frac{k}{m}z_1 - \frac{c}{m}z_2 \end{bmatrix}.$$

which is standard form for numerical integration.

Energy? Note that for the damped oscillator we cannot use energy conservation to check the solution because energy is constantly lost. We could, however, make a plot of the total energy and make sure that it is an ever decreasing (monotonically decreasing) function of time.