10.2 Forcing and resonance

If the world of oscillators was as we have described them so far, especially in Section 9.3, there wouldn't be much to talk about. The undamped oscillators (of which there are none) would be oscillating away and the damped oscillators (all the real ones) would be damped out to no motion. The reason vibrations exist is because they are somehow excited. This excitement is also called *forcing* whether or not it is due to a literal mechanical force.

The most important idea of this section is the following

> If you shake something at about the same frequency at which it naturally oscillates you will eventually get large motions.

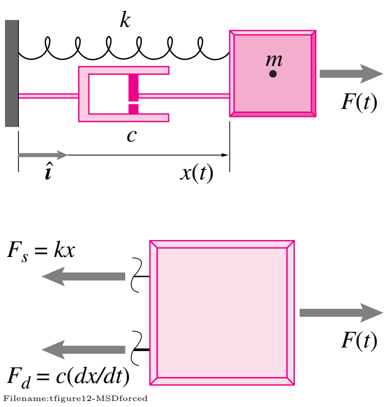

Figure 10.13: A forced mass-spring-dashpot is just a mass held in place by a spring and dashpot but pushed by a force $F(t)$ from some external source.

The rest of the section is largely a fleshing out of this idea.

The simplest example of a 'forced' harmonic oscillator is the mass-spring-dashpot system with an additional mechanical force applied to the mass. See fig. 10.14. Most of this section will be a study of this system. The governing equation for a forced damped oscillator can be derived from the free body diagram as follows, where vector notation helps keep the signs right:

$$\begin{aligned} \sum \vec{F}_i &= m\vec{a} \\ -F_s\hat{\imath} - F_d\hat{\imath} + F(t)\hat{\imath} &= ma\hat{\imath} \\ \{\,(-kx - c\dot{x} + F(t))\,\hat{\imath} &= m\ddot{x}\hat{\imath}\,\} \\ \{\}\cdot\hat{\imath} \quad \Rightarrow \qquad -kx - c\dot{x} + F(t) &= m\ddot{x} \end{aligned}$$

which is often re-arranged as

$$m\ddot{x} + c\dot{x} + kx = F(t). \qquad (10.25)$$

When $F(t) = 0$, there is no forcing and the governing equation reduces to that of the un-forced damped harmonic oscillator, eqn. (10.2).

Equivalent ways to force an oscillator

There are many ways to "force" a system that all lead to the same forced-oscillator equation.

1. With a literal force as in fig. 10.14, shown again in fig. 10.15a.
2. By shaking the support, as in fig. 10.15b.
3. By displacing one end of the spring, but not the dashpot as in fig. 10.15c.
4. By displacing one end of the dashpot, but not the spring as in fig. 10.15d.

5. By displacing a second mass attached to the first with a motor that controls relative position, as in fig. 10.15e.

That these four systems all lead to the same governing equation follows from drawing free body diagrams, applying momentum balance, and collecting terms to match the form eqn. (10.25). Note that the meaning of some of the terms in the forced-oscillation equation is different for each system.

Types of forcing

In general this or that machine or structure could be forced in any number of complicated ways. But there are two special forcings of most common engineering interest:

- $F(t) = F_0$ (Constant force), and
- $F(t) = F \cos \lambda t$ (sinusoidal forcing).①

Constant force idealizes situations where the force doesn't vary much as due say, to gravity, a steady wind, or sliding dry friction. Sinusoidally varying forces are used to approximate oscillating forces as caused, say, by a vibrating support or earthquakes. Forces that are not sinusoidal can be thought of as sums of sine waves thus, in some sense, by knowing how a structure responds to sinusoidal forcing, at various frequencies, you know how it responds to all possible forcings②. Lets look at each of these two cases in detail.

Forcing with a constant force

The case of constant forcing is both common and easy to analyze, so easy that it is often ignored (see fig. 9.27 on page 463). If $F(t) = F_0 =$ constant, then the general solution of eqnrefforcedODE for $x(t)$ is the same as the unforced case but with a constant added. The constant is F_0/k. The usual way of accommodating this case is to describe a new equilibrium point at $x = F_0/k$ and to pick a new deflection variable that is zero at that point. If we pick a new variable z defined as $z = x - F_0/k$, then substituting into eqn. (10.25) we get

$$m\ddot{z} + c\dot{z} + kz = 0, \qquad (10.26)$$

which is the unforced oscillator equation. That is, constant forcing reduces to the case of no forcing if one merely changes what one calls zero to be the place where the mass is in equilibrium, taking account of the spring stretch (or compression) caused by the constant applied force. Thus the solution of the forced equation for x is equivalent to the unforced solution for z:

$$z(t) = x(t) - F_0/k = e^{(-\frac{c}{2m})t}\,(A\cos(\lambda_d t) + B\sin(\lambda_d t)) \qquad (10.27)$$

where $\lambda_d = \sqrt{\left(\frac{c}{2m}\right)^2 - k/m}$, as explained in box 10.2 on page 501.

An alternative approach is to use *superposition*. Here we say $x(t) = x_h(t) + x_p(t)$ where $x_h(t)$ satisfies $m\ddot{x} + c\dot{x} + kx = 0$ and $x_p(t)$ is any solution x_p of $m\ddot{x} + c\dot{x} + kx = F_0$. Any solution you like is called a

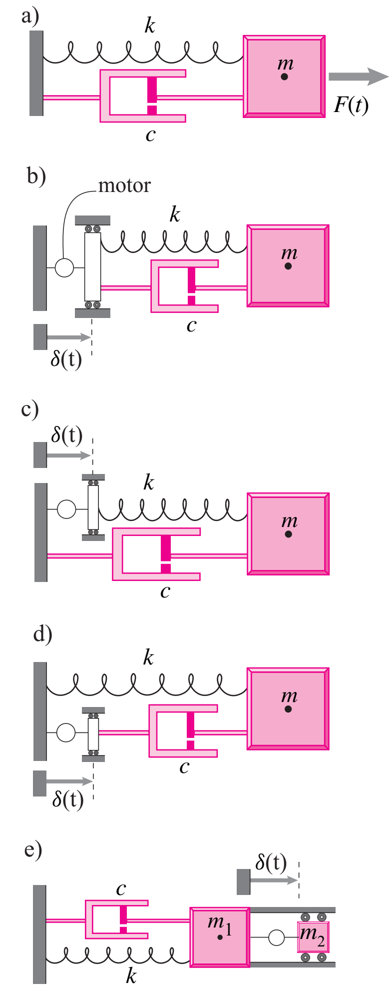

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Figure 10.14: In all cases shown above the same forced oscillator eqn. (10.25) applies. In (a) a literal force is applied. In all the other cases the "forcing" is by a motor that moves something back and forth a distance δ. In (b) the support moves. In (c) and (d) just the spring or just the dashpot end is displaced. In (e) an extra mass is moved relative to the main mass.

①Note again that we use λ (lambda) here instead of the more commonly used ω (omega) because we want to avoid confusion with the magnitude of angular velocity $\omega = |\vec{\omega}|$ which itself could have oscillatory motion. It would be confusing (and bad math) to write $\omega = A \sin \omega t$ with ω having two different meanings in the same equation.

② The best approximation of a function as a sum of sine waves is a Fourier series, a topic you learn in math, advanced physics or linear systems courses.

"particular" solution. One easy solution is $x_p = F_0/k$. So the net solution is $x_p = F_0/k$ plus a solution x_h to the 'homogeneous' equation 10.26.

$$x(t) = \underbrace{e^{(-\frac{c}{2m})t}\left(A\cos(\lambda_d t) + B\sin(\lambda_d t)\right)}_{x_h} + \underbrace{F_0/k}_{x_{p1}} \qquad (10.28)$$

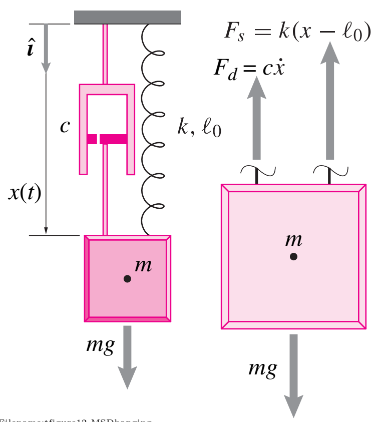

Figure 10.15: A mass hangs from a spring and dashpot. Its distance x from the support is due to the rest length ℓ_0 of the spring, and the stretch of the spring.

Example: Hanging mass.
The mass hanging from the support shown in fig. 10.16 obeys the equation

$$m\ddot{x} + c\dot{x} + kx = \underbrace{k\ell_0 + mg}_{F_0}$$

One particular solution x_p, the easiest one, has the mass hanging still. In this solution, the mass position is the un-stretched length ℓ_0 of the spring plus the stretch of the spring due to gravity, $\Delta x = mg/k$. Because the mass is still in this solution, the dashpot constant c doesn't appear. So

$$x_p = \ell_0 + mg/k.$$

The homogeneous solution x_h is given by (10.27) and the general motion is the sum

$$\begin{aligned} x(t) &= x_p + x_h \\ &= (\ell_0 + mg/k) + e^{-\frac{ct}{2m}}\left(C\cos(\lambda_d t) + D\sin(\lambda_d t)\right) \end{aligned}$$

where C and D are constants determined by the initial conditions. For any initial condition and corresponding values of A and B, the motion eventually decays to the stationary particular solution with the mass hanging still (because the exponentials go to zero as $t \to \infty$).

Forcing with a sinusoidally varying force

The motion resulting from sinusoidal forcing is of central interest in vibration analysis. In this case we imagine that $F(t) = F\cos pt$ where F is the amplitude of forcing and p is the angular frequency of the forcing. Note, we could just as well use $F(t) = F\sin pt$ for the forcing, sin and cos are both sinusoidal forcings.

The general solution of equation 10.25 is given by the sum of two parts. One is the general solution of equation 10.2, $x_h(t)$, and the other is *any* solution of equation 10.25, $x_p(t)$. The solution $x_h(t)$ of the damped oscillator equation 10.2 is called the 'homogeneous' or 'complementary' solution. Any solution $x_p(t)$ of the forced oscillator equation 10.25 is called a 'particular' solution.

We already know the solution $x_h(t)$ of the undamped governing differential equation 10.2. This solution is equation 10.3, 10.5, or 10.6, depending on the values of the mass, spring and damping constants. So the new problem is to find any solution to the forced equation 10.25. The easiest way to solve this (or any other) differential equation is to make a fortuitous guess (you may learn other methods in your math classes). In this case with

$$F(t) = F\cos(pt)$$

we make the guess that

$$x_p(t) = A\cos(pt) + B\sin(pt). \tag{10.29}$$

Basically this guess says "If you shake something with a sine wave it will probably move as a sine wave. But who knows the amplitude or phase?" Plugging this guess into the forced oscillator equation (10.25) we find values for A and B in box 10.2 on page 523.

Alternatively, a sum of sine waves can be written as a cosine wave (or sine wave) that has been shifted in phase as (see box 9.2 on page 460)

$$x_p(t) = A_0\cos(pt - \phi),$$

The value of forced amplitude is simply $A_0 = A^2 + B^2$ and is also given in terms of m, c, k, p and F in box 10.2. The forced amplitude A_0 is the central subject of this section. It answers the question 'How big are the oscillations when you shake something.?' Because the formula for A_0 is admittedly a mess, the answer is often given in a plot③. The general solution, therefore, is

$$x(t) = x_h(t) + x_p(t). \tag{10.30}$$

③Another more important reason that a plot is used is that often in a physical system one can measure the vibrations while never knowing a detailed accurate set of differential equations which would describe the system accurately.

The homogeneous solution $x_h(t)$, the motion of the unforced system, is just decaying oscillations and is usually not of primary interest in vibrating systems. The particular solution $x_p(t)$ is steady oscillations. These oscillations are of central interest. In particular most often in engineering one wants these oscillations to be big or small.

Example: MEMs devices.
One general type of "**M**icro **E**lectronic **M**achine" consists of, basically, a vibrating beam. A beam with an effective mass 50μ gm and effective stiffness of $k = 500\,\text{N/m} = 5\mu\,\text{N}/\mu\,\text{m}$ has

$$\lambda_n = \sqrt{\frac{k}{m}} = \sqrt{\frac{5\,\text{N/m}}{50\cdot 10^{-9}\,\text{kg}}} = \sqrt{\frac{\frac{500\text{N}}{\text{m}}\left(\frac{1\,\text{kg m}/s^2}{1\text{N}}\right)}{50\cdot 10^{-6}\,\text{kg}}} = \sqrt{10^{10}}\,\text{s}^{-1} = 10^5/\,\text{s}$$

which corresponds to a frequency of $\lambda_n/2\pi \approx 15.7\text{k hz}$. That is, such a MEMs device would be a good receiver (or 'resonator') for 15.7k hz ultra-sonic vibrations. In this case resonance is useful to make the sensor sensitive.

The size of the oscillations scales with the size of the forcing F (this proportionality is known as 'linearity') and also depends on all the parameters m, c, k and p.

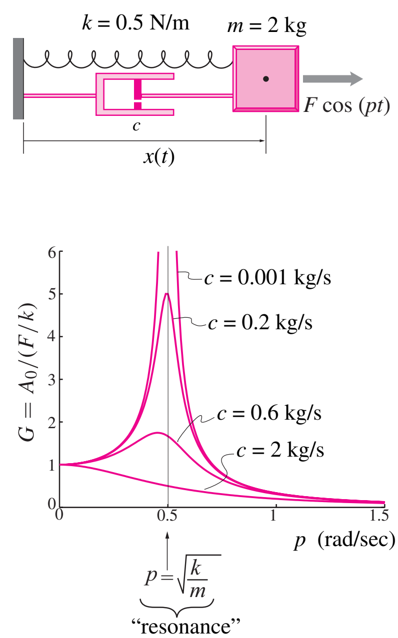

Figure 10.16: Amplitude of oscillation vs forcing frequency for various dampings. Each curve shows the gain G vs the forcing frequency for a fixed damping. Note that when the damping is small ($c \ll 1$) and the forcing is close to the natural frequency of vibration $p \approx \lambda_n$ there is a 'resonant' peak in the amplitude of the response. The smaller the damping the higher and narrower is this peak. For very high damping the peak is at a slightly lower frequency. The mass-spring-dashpot system shown was used to generate the plots using the formulas from box 10.2 on page 523.

Frequency response and resonance

One way to show a structure's sensitivity to oscillatory loads is by a *frequency response* curve fig. 10.17. One curve shows the amplitude of vibrations vs the forcing frequency. The main idea of this section, resonance, shows as a peak in the frequency-response curve near the natural frequency $\lambda_n = \sqrt{k/m}$.

Recall that the natural frequency λ_n is the unforced frequency of undamped oscillation. The damped natural frequency λ_d, the frequency of decaying oscillations with damping present, is slightly slower (see, e.g., eqn. (10.3) on page 501.). The resonant frequency λ_{res}, the frequency of forcing for which the amplitude of motion is maximum (eqn. (10.2) on page 523), is slightly lower still. But, especially when the damping is low, there is only a small difference between the natural frequency, the damped frequency and the resonant frequency. So, in common language and engineering practice they are usually treated as one and the same.

In summary, the frequency response curve has a peak with forcing near to, but not exactly at, the natural frequency of unforced and undamped motion. But most engineers can reasonably assume, even though its not exactly true, that resonance occurs when the forcing frequency *is* the natural vibration frequency.

④**Golden Gate bridge cables.** On a walk a few years ago one of us noted that the vertical cables on the Golden Gate bridge could be induced to oscillate quite visibly if pushed by a person at the right frequency (about 0.5 hz). One cable, maybe 100 tons of steel, was nicely going back and forth about a half a meter. A police car pulled up an stopped. Through the megaphone the officer authoritatively and sternly threatened "If you break it you have to pay for it."

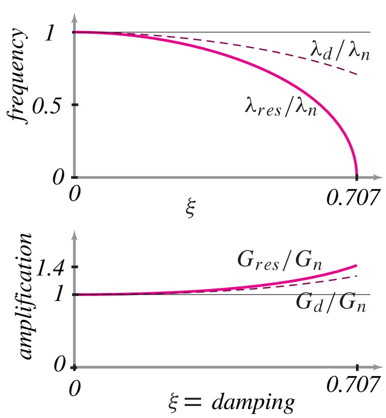

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Figure 10.17: **Natural, damped and resonant frequencies.** The upper dashed plot shows the ratio of damped to natural frequency vs forcing frequency ratio ξ. The solid curve shows how the resonant frequency varies from the natural frequency as a function of frequency ratio. The lower plot shows how the amplification $G = A_0/(F/k)$ is nearly the same at the natural, damped and resonant frequencies. That the peak in the $c = 0.6\text{kg/s}$ curve of fig. 10.17 is slightly to the left of $p = \lambda_n = \sqrt{k/m}$ shows as $\lambda_{\text{res}}/\lambda_n$ being slightly below 1 for $\xi \approx 0.3$. For low damping an engineer can treat the natural, damped and resonant frequencies as equal. Similarly the amplification when forced at the natural frequency is very close to the amplification when forcing is at the damped or at the resonant frequency.

Resonance is good and bad

Sometimes an engineer studies vibrations with the hope of minimizing them, sometimes with the hope of maximizing them. Resonance is sometimes the problem and sometimes the solution.

Resonant vibrations are usually undesirable in machinery or cars. The vibrations can lead to large stresses, undesirable motions, or unpleasant sounds. A building resonating to earthquake vibrations may be more likely to fall down.

On the other hand, nuclear Magnetic Resonance imaging is used for medical diagnosis. In the old days, the resonant excitation of a clock pendulum was used to keep time. The resonance of quartz crystals is used to time most watches now-a-days. Self excited resonance is what makes musical instruments have such clear pitches. And resonant vibrations are used to give a larger signal in micro-mechanical sensors. In the electrical domain, radio tuners depend on resonance to pick out just one radio band.

Other systems

Most machines and structures are not exactly a point mass moving in one direction and constrained by a single spring and single dashpot. On the other hand, almost all machines have mass, elastic give, and some dissipation when they move. So most machines have natural oscillations after they are banged or disturbed somehow. And so most structures and machines can be shaken to large motions if the appropriate (or inappropriate, depending on your aims) frequency of force is applied④.

So the concepts introduced here for a single mass-spring-dashpot system apply to much more complex machines and structures. In particular, have natural vibration frequencies and they shake a lot (resonate) if forced at near those frequencies.

Experimental measurement

Because no real thing of interest is exactly a single mass-spring-dashpot the ideas of vibrations analysis are often not expressed in terms of $(m, c$ and $k)$.

Rather, the more broad ideas of natural frequency, frequency response, and resonance are considered on their own. Using either a large-scale computer model (say a 'finite-element' model) or measurement of the physical system itself, one can draw a frequency-response curve like fig. 10.17 on page 517.

Here's how. First you apply a sinusoidal force to the structure at the point of interes, say $F = F\cos(pt)$. Then you measure the motion of a part of the structure of interest. You might instead measure a strain or rotation, but for definiteness let's assume you measure the displacement of some point on the structure δ.

If the structure is linear and has some damping, the eventual motion of the structure will eventually be a sinusoidal oscillation. In particular, you will measure that

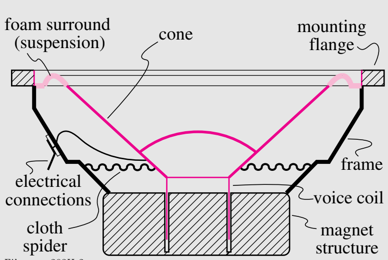
$$\delta = A_0 \cdot \cos(pt - \phi). \qquad (10.32)$$

If you had applied half as big a force, you would have measured half the displacement, still assuming the structure is linear, so the ratio of the displacement to the force A_0/F is independent of the size of the force F. Let's define:

$$G = \frac{A_0}{F}$$

That is, the amplification gain G is the ratio of the amplitude of the displacement sine wave to the amplitude of the forcing sine wave. Plotting p on the x axis and G on the y axis, this experiment gives one point on the frequency

10.1 A Loudspeaker cone is a forced oscillator.



Cross-sectional view of a speaker.

A speaker, similar to the ones used in many home and auto speaker systems, is one of many devices which may be conveniently modeled as a one-degree-of-freedom mass-spring-dashpot system. A typical speaker has a paper or plastic cone, supported at the edges by a roll of plastic foam (the surround), and guided at the center by a cloth bellows (the spider). It has a large magnet structure, and (not visible from outside) a coil of wire attached to the point of the cone, which can slide up and down inside the magnet. (The device described above is, strictly speaking, the speaker driver. A complete speaker system includes an enclosure, one or more drivers, and various electronic components.) When you turn on your stereo, the amplifier forces a current through the coil in time with the music, causing the coil to alternately attract and repel the magnet. This rapid oscillation of attraction and repulsion results in the vibration of the cone which you hear as sound.

In the speaker, the primary mass is comprised of the coil and cone, though the air near the cone also contributes as 'added mass.' The 'spring' and 'dashpot' effects in the system are due to the foam and cloth supporting the cone, and perhaps to various magnetic effects. Speaker system design is greatly complicated by the fact that the air surrounding the speaker must also be taken into account. Changing the shape of the speaker enclosure can change the effective values of all three mass-spring-dashpot parameters. (You may be able to observe this dependence by cupping your hands over a speaker (gently, without touching the moving parts), and observing amplitude or tone changes.) Nevertheless, knowledge of the basic characteristics of a speaker (e.g., resonance frequency), is invaluable in speaker system design.

Our approximate equation of motion for the speaker is identical to that of the ideal mass-spring-dashpot above, even though the forcing is from an electromagnetic force, rather a than a direct mechanical force:

$$m\ddot{x} + c\dot{x} + kx = F(t) \text{ with } F(t) = \alpha i(t) \qquad (10.31)$$

where $i(t)$ is the electrical current flow through the coil in amps, and α is the electro-mechanical coupling coefficient, in force per unit current.

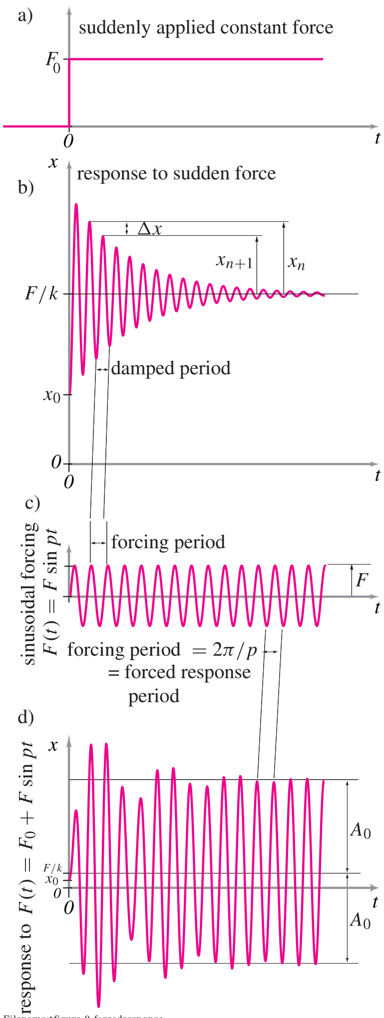


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Figure 10.18: **Transient response.** (a) shows a suddenly applied force F_0. The response (b) to this force is a motion that starts at the initial position x_0 ($x_0 > 0$ in this illustration) and then oscillates about the new equilibrium F_0/k. The motion is identical to unforced motion, but offset. Thus it can be used to evaluate the rate of decay of oscillation and the damped period of oscillation. (c) A sinusoidal forcing causes. (d) the response if the mass is released at $x = x_0$ and suddenly both a constant force F and a sinusoidal force $F \sin pt$ are applied. The motion eventually settles into a sinusoidal oscillation at the forcing frequency (which is a little longer period than the damped oscillation in this illustration) with amplitude A_0.

response curve. Repeating for a range of forcing frequencies one can plot up the frequency response $G = G(p)$.

Example: Shake table for earthquake response.
One way to get a frequency response curve for a building is to put a scale model on a "shake table". The base is then moved sinusoidally through a range of frequencies and the motion of the model is observed. This way one can find peaks in the frequency-response curve. These are frequencies that, to the extent they are prevalent in a feared earthquake, are likely to cause damage.

Transient response

As discussed, the full solution of eqn. (10.25) with forcing $F(t) = F_0 + F \cos pt$ is the sum of three terms

$$x(t) = x_h + x_{p1} + x_{p2}$$

The first of these has decaying oscillations, the second is a constant, and the third has steady oscillations. When added up the motion can look quite complicated, as seen in fig. 10.19. The main point is that after some initial complicated transient the motion eventually decays to steady oscillations ($x_{p2}(t) = A_0(\cos pt - \phi)$) plus an offset ($x_{p1} = F/k$).

The vocabulary of forced oscillations

Forced oscillations are so important and common that there is a specialized vocabulary for many of the terms and collections of commonly appearing terms. Here is a list, starting with the terms you know well.

m = the **mass** of the particle that is oscillating. For more complicated systems the mass m may represent an "effective" or "equivalent" mass.

c = the **damping coefficient**. c is used to describe the viscous drag, the resistance to motion $F_d = -c\dot{x}$.

k = the **spring constant**. k describes the elastic restoring "spring" force $F_s = -kx$.

F = the **forcing amplitude** for a sinusoidally varying applied force $F(t) = F \sin pt$ or $F(t) = F \cos pt$ or $F(t) = A \sin pt + B \cos pt$ with $F \equiv \sqrt{A^2 + B^2}$.

p = the **forcing frequency**. Some books will use the symbol ω for the forcing frequency.

The rest of the quantities below are completely determined by the quantities above (m, c, k, F and p).

$\lambda_n \equiv \sqrt{k/m}$ is the **natural frequency**. This is the frequency of oscillation if there is neither forcing nor damping. In that case $x(t) = A \cos \lambda_n t + B \sin \lambda_n t$. Many books use ω_n for the natural frequency.

$c_{crit} = 2 * \sqrt{km}$ is the **critical damping coefficient**. The relation of the actual damping c to the critical damping c_{crit} tells you whether a system is over-damped ($c > c_{crit}$ $\Rightarrow$ decay to equilibrium, when unforced, that is exponential) or under-damped ($c < c_{\text{crit}} \Rightarrow$ decay to equilibrium, when unforced, that is oscillatory). See Fig. 10.3 on page 500. Sometimes c_{crit} is more simply written as c_c or c_{cr}.

$\xi \equiv c/c_{\text{crit}}$ is the **damping ratio**. The single number ξ ('ksee') tells you if a system is over damped ($\xi > 1$) or underdamped ($\xi < 1$).

$r \equiv p/\lambda_{\text{n}} = p/\sqrt{k/m}$ is the **frequency ratio** . If $r > 1$ then the forcing is faster than the frequency of natural unforced vibrations. If $r < 1$ then the forcing is slower than the natural vibrations.

A_0 = the **response amplitude** . When a steady oscillatory force is applied the motion is eventually oscillatory. The amplitude of the motions is A_0, as in $x = A_0 cos(pt - \phi)$ with

$$A_0 = (F/k)/\sqrt{(2\xi r)^2 + (1 - r^2)^2)}.$$

$G \equiv A_0/(F/k)$ is the **gain** or **amplification**. G is the ratio of the eventual amplitude of the oscillator to the response that would occur if the same force was applied at zero frequency. It is the response amplitude scaled by the displacement that would occur if the same force was applied to a spring.

$\lambda_{\text{res}} = \lambda_{\text{n}}\sqrt{1 - 2\xi^2}$ is the **resonant frequency.** λ_{res} (also called λ_r or ω_r) is the frequency such that if $p = \lambda_{\text{res}}$ the amplification gain G is maximum. The resonant frequency is the frequency at which you force a system to get the biggest motions. The resonant frequency λ_{res} is rather close to the natural frequency λ_{n} in systems with small damping ratios. And these are also the systems that are prone to resonant vibrations.

λ_{d} is the **damped natural frequency** . If an underdamped system is released from rest it oscillates as the motions decay. The frequency of these oscillations is

$$\lambda_{\text{d}} = \lambda_{\text{n}}\sqrt{1 - \xi^2}.$$

The frequency λ_{d} of damped oscillations is a shade slower than the frequency λ_{n} of oscillation of the same system with no damping. When damping is small the natural frequency λ_{n}, the damped frequency λ_{d} and the resonant frequency λ_r are all close to each other (See fig. 10.17a).

$G_{\text{n}}, G_{\text{res}}$ & G_{d} are the amplification **gains** (see G above) when forcing is at the natural, the resonant and the damped natural frequency respectively ($p = \lambda_{\text{n}}, \lambda_{\text{res}}$ & λ_{d}, see above). G_{res} is the biggest of these by definition. But it is not actually much bigger than G_n or G_d. These gains can be calculated using the formulas for G and A_0 above. They are plotted on fig. 10.17b.

D is the **logarithmic decrement**. D measures the rate of decay of unforced ($F = 0$) oscillations. The experimental definition, derivable from a graph of the motion, is

$$D = \ln(\frac{x_n}{x_{n+1}}).$$

In terms of m, c and k the logarithmic decrement is $D = \frac{cT}{2m} = 2\pi\xi$, as derived on page 501. If there is little damping, c is small ($\xi \ll 1$) and $D \approx (x_n - x_{n+1})/x_n$ is the fractional decrease in amplitude per oscillation. If $D = .1$ then each oscillation is about 10% smaller in amplitude than the previous one.

Q is the **quality factor**. For the mass-spring-dashpot system it is another way of describing the rate of decay of unforced oscillations.

$$\begin{aligned} Q &\equiv 2\pi(\text{energy of oscillator})/(\text{energy lost per cycle}) \\ &= 2\pi x_n^2/(x_n^2 - x_{n+1}^2) \\ &\approx \pi/D = 1/(2\xi) \quad \text{(for small damping)} \end{aligned}$$

The π in the definition of Q makes it so there is no π in the formula for the quality factor Q in terms of the damping ratio ξ. Note that, so long as damping is small, ξ, D and Q can each be found approximately from the other. A system with low damping ($\xi \ll 1$) has high quality ($Q \gg 1$) and slowly decaying oscillations and hence a small logarithmic decrement ($D \ll 1$).

10.2 Solution of the forced oscillator equation

The main equation for understanding forced oscillations is:

$$m\ddot{x} + c\dot{x} + kx = F_0 + F\cos pt.$$

Because the equation is linear we look for a solution which is the sum of three terms

$$x(t) = x_h + x_{p1} + x_{p2}$$

where x_h is the homogeneous solution from Eqns 10.3 - 10.6 on page 501, depending on whether the system is underdamped (oscillatory decay), critically damped or over damped (non-oscillatory exponential decay). x_{p1} is a particular solution for the constant forcing F_0. $x_{p1}(t)$ was found in eqn. (10.28) on page 516 to be, simply, $x_{p1} = F_0/k$.

The last part of the solution, finding an x_{p2} for the forcing term $F\cos pt$ is found by guessing

$$x_{p2} = A\cos pt + B\sin pt.$$

When this guess is plugged into the equation

$$m\ddot{x} + c\dot{x} + kx = F\cos pt$$

every term is either a multiple of $\sin pt$ or $\cos pt$. Thus we get

$$\{\text{A collection of constants}\}\cos pt + \{\text{Another collection}\}\sin pt = 0$$

The only way a sum of a sine wave and cosine wave can be zero for all time is for both coefficients to be zero. Setting the two collections of constants above both to zero gives two simultaneous equations for the unknowns A and B in terms of m, c, k and p. These can be solved to give

$$A = \frac{(F/k)\left(1 - \frac{p^2}{(k/m)}\right)}{\left(\frac{c^2}{km}\right)\left(\frac{p^2}{k/m}\right) + \left(1 - \frac{p^2}{(k/m)}\right)^2},$$

and

$$B = \frac{(F/k)(cp/k)}{\left(\frac{c^2}{km}\right)\left(\frac{p^2}{k/m}\right) + \left(1 - \frac{p^2}{(k/m)}\right)^2}.$$

So we have found the particular solution for forcing with $F(t) = F\cos pt$, using A and B above, as

$$x_{p2} = A\cos pt + B\sin pt. \tag{10.33}$$

An alternative form for the solution is

$$x_{p2}(t) = A_0\cos(pt - \phi), \tag{10.34}$$

for which we can find the constants A_0 and ϕ using the trig identity $\cos(\theta - \phi) = \cos\theta\cos\phi + \sin\theta\sin\phi$ described in box 9.2 on page 460. Applying this identity to the solution above we find the object of central interest, the forced amplitude

$$A_0 = \sqrt{(A^2 + B^2)} = \frac{F/k}{\sqrt{\left(\frac{c^2}{km}\right)\left(\frac{p^2}{k/m}\right) + \left(1 - \frac{p^2}{(k/m)}\right)^2}} \tag{10.35}$$

and also the phase angle

$$\phi = \tan^{-1}\left(\frac{B}{A}\right) = \tan^{-1}\left(\frac{cp/k}{\left(1 - \frac{p^2}{(k/m)}\right)}\right). \tag{10.36}$$

All of the expressions above can be somewhat simplified if write them in terms of the *frequency ratio* $r = p/\lambda_n = p/\sqrt{k/m}$ and damping ratio $\xi = c/c_{crit} = c/2\sqrt{km}$ (The frequency ratio, damping ratio and some more specialized vibration words are defined on page 520.). Using these dimensionless quantities, the values of the constants in the solution $x_{p2}(t)$, namely eqn. (10.33) or eqn. (10.34), are:

$$A = \frac{(F/k)(1 - r^2)}{4\xi^2 r^2 + (1 - r^2)^2},$$

$$B = \frac{(F/k)(2\xi r)}{4\xi^2 r^2 + (1 - r^2)^2},$$

$$A_0 = \sqrt{(A^2 + B^2)} = \frac{F/k}{\sqrt{(2\xi r)^2 + (1 - r^2)^2}},$$

and

$$\phi = \tan^{-1}\left(\frac{B}{A}\right) = \tan^{-1}\left(\frac{2\xi r}{1 - r^2}\right).$$

These constants are for the particular forced solution $x_{p2}(t)$ of eqn. (10.33) or eqn. (10.34). Again, most important in all of this is the amplitude A_0 of the forced response. As you can see, the bottom of the fraction for A_0 gets quite small for small damping ($\xi \ll 1$) if the frequency ratio r is close to 1. That is,

> *the amplitude is big if the forcing is close to the natural frequency.*

Resonant frequency

In detail, the frequency at which the vibration amplitude A_0 is maximum is not exactly the unforced undamped natural frequency $\lambda_n = \sqrt{k/m}$. The resonant frequency λ_{res} is found by maximizing A_0 with respect to $r = \lambda/\lambda_n$. Setting $dA_0/dr = 0$ and solving for r we find

$$r_{res} = \sqrt{1 - 2\xi^2} \quad \Rightarrow \quad \lambda_{res} = \lambda_n\sqrt{1 - 2\xi^2} \tag{10.37}$$

The ratio of λ_{res}/λ_n is plotted on fig. 10.17 on page 517. Also plotted is the ratio of A_0 at resonance to A_0 if forcing is at the natural frequency. The morals are that a) for small damping the natural frequency and resonant frequency are very close, and b) for all dampings, there is little error in calculating the amplitude of the maximum vibration response by approximating resonance as being at the natural frequency. Even when resonance is barely a viable concept, for systems that are critically damped, the error is only 40%.

Similarly one might think the damped natural frequency

$$\lambda_d = \lambda_n\sqrt{1 - \xi^2}$$

would be a better approximation to the resonant frequency. Actually, its about half way between the natural and resonant frequencies, as can be seen also on fig. 10.17.