10.3 Normal modes

To read this section you need to know the linear algebra concepts of eigenvalues and eigenvectors. Systems with many moving parts often move in complicated ways. Consider the two mass system shown in fig. 10.30. By drawing free body diagrams and writing linear momentum balance for the two masses we can write the equations of motion in matrix form (see eqn. (9.44)) as

$$[M]\ddot{\boldsymbol{x}} + [K]\boldsymbol{x} = \boldsymbol{0}$$

where

$$[M] = \begin{bmatrix} m & 0 \\ 0 & m \end{bmatrix} \quad \text{and} \quad [K] = \begin{bmatrix} -2k & k \\ k & -2k \end{bmatrix}.$$

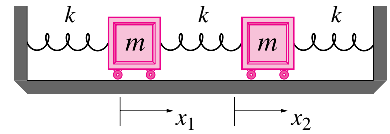

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Figure 10.29: A two mass system. We define x_1 and x_2 so that the system is in equilibrium when $x_1 = x_2 = 0$.

Example: Complicated motion.
If we put the initial condition

$$\boldsymbol{x}_0 = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad \text{and} \quad \boldsymbol{v}_0 = \begin{bmatrix} 0 \\ 0 \end{bmatrix},$$

we get the motion shown in fig. 10.31a. Both masses move in a complicated way and not synchronously with each other.

On the other hand, all such systems, if started in just the right way, will move in a simple way.

Example: Simple motion: a normal mode.
If we put the initial condition

$$\boldsymbol{x}_0 = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \quad \text{and} \quad \boldsymbol{v}_0 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

we get the motion shown in fig. 10.31b. Both masses move in a simple sine wave, synchronously and *in phase* with each other.

That this system has this simple motion is intuitively apparent. If both of the equal masses are displaced equal amounts both have the same restoring force. So both move equal amounts in the ensuing motion. And nothing disturbs this symmetry as time progresses. In fact the frequency of vibration is exactly that of a single spring and mass (with the same k and m).

A given system can have more than one such simple motion.

Example: Another normal mode.
If we put the initial condition

$$\boldsymbol{x}_0 = \begin{bmatrix} 1 \\ -1 \end{bmatrix} \quad \text{and} \quad \boldsymbol{v}_0 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

we get the motion shown in fig. 10.31c. Both masses move in a simple sine wave, synchronously and exactly *out of phase* with each other. Being exactly out of phase is actually a form of being exactly in phase, but with a negative amplitude.

This motion is also intuitive. Each mass has restoring force of $3k\Delta x$. One k from a spring at the end and $2k$ because each mass experiences a spring with half the length (and thus twice the stiffness) in the middle (because the middle of the middle spring doesn't move in this symmetric motion).

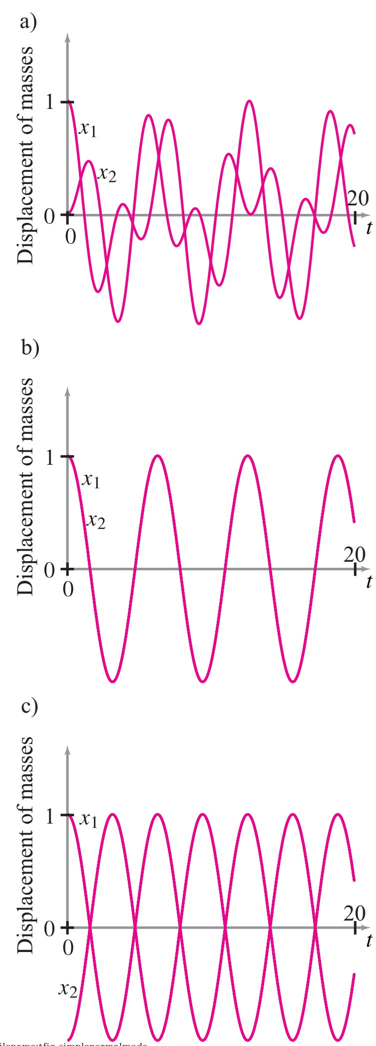

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Figure 10.30: Motions of the masses from fig. 10.30 for three different initial conditions, all released from rest ($v_1 = v_2 = 0$)
a) $x_1 = 1,\ x_2 = 0$,
b) $x_1 = 1,\ x_2 = 1$, and
c) $x_1 = 1,\ x_2 = -1$.

The system above is about the simplest for demonstration of *normal mode* vibrations. But more complicated elastic systems always have such simple normal mode vibrations.

All elastic systems with mass have *normal mode* vibrations in which all masses

- have simple harmonic motion
- with the same frequency as all the other masses, and
- exactly in (or out) of phase with all of the other masses

Thus the first and second normal modes from fig. 10.31b,c can be written as

$$\underbrace{\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} \cos\lambda_1 t \\ \cos\lambda_1 t \end{bmatrix}}_{\text{First normal mode}} \quad \text{and} \quad \underbrace{\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} \cos\lambda_2 t \\ -\cos\lambda_2 t \end{bmatrix}}_{\text{Second normal mode}}$$

where, by the physical reasoning in the examples we know that $\lambda_1 = \sqrt{k/m}$ and $\lambda_2 = \sqrt{3k/m}$. We could equally well have used the sine function instead of cosine.

Superposition of normal modes

Note that the governing equation (eqn. (10.3)) is 'linear' in that the sum of any two solutions is a solution. If we add the two solutions from fig. 10.31b,c we have a solution. And if we divide that sum by two we get a solution. And not just any solution, but the solution in fig. 10.31a. The top curve is the sum of the bottom two divided by two (The curves for $x_1(t)$ and $x_2(t)$ need to be added separately).

For more complicated systems it is not so easy to guess the normal modes. Most any initial condition will result in a complicated motion. Nonetheless the concept of normal modes applies to any system governed by the system of equations (eqn. (10.3)):

$$[M]\ddot{\boldsymbol{x}} + [K]\boldsymbol{x} = \boldsymbol{0}.$$

Any collection of springs and masses connected any which way has normal mode vibrations. And because elastic solids are the continuum equivalent of a collection of springs and masses, the concept applies to all elastic structures. Here are the basic facts

- An elastic system with n degrees of freedom has n independent normal modes.
- In each normal mode i all the points move with the same angular frequency λ_i and exactly in phase.
- Any motion of the system is a superposition of normal modes (a sum of motions each of which is a normal mode).

Example: Musical instruments
The pitch of a bell is determined by that normal mode of the bell that has the lowest natural frequency. Similarly for violin and piano strings, marimba keys, kettle drums and the air-column in a tuba.

A recipe for finding the normal modes of more complex systems is given in box 10.3 on page 533.

Normal modes and single-degree-of-freedom systems

Any complex elastic system has simple normal mode motions. And all motions of the system can be represented as a superposition of normal modes. Hence sometimes we can think of every system as if it *is* a single degree of freedom system. For example, if a complex elastic system is forced, it will resonate if the frequency of forcing matches any of its normal mode (or natural) frequencies.

The math of, and how to find, normal modes

Consider a collection of n masses connected by springs whose motions are governed by eqn. (10.3)

$$[M]\ddot{\boldsymbol{x}} + [K]\boldsymbol{x} = \boldsymbol{0},$$

where the positions of the masses are $\boldsymbol{x} = \boldsymbol{x}(t) = [x_1(t), x_2(t), ..., x_n(t)]'$. The matrices $[M]$ and $[K]$ have to do with the masses and the network of springs, respectively. At this point in the book the examples are masses in a line, but the concepts are more general.

How to find a solution. The approach used by the professionals is to *guess* that there are "normal mode" solutions and then see if they are. A normal mode solution, with all masses moving sinusoidally and synchronously, is

$$\boldsymbol{x} = \begin{bmatrix} V_1 \cos \lambda t \\ V_2 \cos \lambda t \\ \vdots \end{bmatrix} = \boldsymbol{V} \cos \lambda t.$$

Upper case bold $\boldsymbol{V}$ (to distinguish it from lower case velocity) is a list of constants $[V_1, V_2, ...]'$. We could have used sin just as well as cos for our guess. Now we plug our guess into the governing equations to see if it is a good guess:

$$\begin{aligned} [M]\ddot{\boldsymbol{x}} + [K]\boldsymbol{x} &= \boldsymbol{0} \\ [M]\frac{d^2}{dt^2}\{\boldsymbol{V}\cos\lambda t\} + [K]\{\boldsymbol{V}\cos\lambda t\} &= \boldsymbol{0} \\ -\lambda^2[M]\boldsymbol{V}\cos\lambda t + [K]\boldsymbol{V}\cos\lambda t &= \boldsymbol{0} \\ \left\{-\lambda^2[M]\boldsymbol{V} + [K]\boldsymbol{V}\right\}\cos\lambda t &= \boldsymbol{0}. \end{aligned}$$

This equation has to hold true for all t therefore the constant column vector inside the brackets $\{\}$ must be zero:

$$-\lambda^2[M]\boldsymbol{V} + [K]\boldsymbol{V} = \boldsymbol{0}$$
$$\left[-\lambda^2[M] + [K]\right]\boldsymbol{V} = \boldsymbol{0}$$

The matrix $[M]$ is usually invertible. If $[M]$ is diagonal its inverse is $[M]$ with each element replaced by its reciprocal. Assuming $[M]^{-1}$ exists we can multiply through by $[M]^{-1}$ to get:

$$[M]^{-1}[K]\boldsymbol{V} = \lambda^2\boldsymbol{V},$$

where we used that $[M]^{-1}[M] = [1]$ = the identity matrix, and that $[1]\boldsymbol{V} = \boldsymbol{V}$. Defining the product $[B] = [M]^{-1}[K]$ and substituting we get the classic eigenvalue problem:

$$[B]\boldsymbol{V} = \lambda^2\boldsymbol{V}. \tag{10.45}$$

There is a lot to know about eqn. (10.45). Its a famous equation. Equation (10.45) says that $\boldsymbol{V}$ is a vector that, when multiplied by $[B]$ gives itself back again, multiplied by a constant. For the special vector $\boldsymbol{V}$, being multiplied by the matrix $[B]$ is equivalent to being multiplied by the scalar λ^2.

Given $[B]$ there are generally n linear independent *eigen vectors* $\boldsymbol{V}^1, \boldsymbol{V}^2, ...\boldsymbol{V}^n$ with associated *eigen values* $\lambda_1^2, \lambda_2^2, ..., \lambda_n^2$. Note, $[B]$ is generally not symmetric.

In the case of our vibration problem the eigen vectors are called *modes* or *eigen modes* or *mode shapes* or *normal modes*.

Recipe for finding normal modes

Given the matrices $[M]$ and $[K]$ proceed as follows.

- Calculate $[B] = [M]^{-1}K$
- Use a math computer program to find the eigenvalues and eigenvectors of $[B]$, call these $\boldsymbol{V}^i$ and λ_i^2. Usually this is a single command, like:

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eig(B)
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- For each i between 1 and n write each normal mode as $\boldsymbol{x}(t) = \boldsymbol{V}\cos(\lambda_i t)$ or as $\boldsymbol{x}(t) = \boldsymbol{V}\sin(\lambda_i t)$

For example, if

$$[M] = \begin{bmatrix} m & 0 \\ 0 & m \end{bmatrix} \quad \text{and} \quad [K] = \begin{bmatrix} -2k & k \\ k & -2k \end{bmatrix}.$$

then, for any values of k and m, the computer will return for the eigen values and eigenvectors of $[B] = [M]^{-1}[K]$:

$$\boldsymbol{V}^1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \text{with } \lambda_1^2 = k/m \text{ and } \boldsymbol{V}^2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix} \text{with } \lambda_2^2 = 3k/m$$

Why are they called 'normal' modes?: Another recipe

The math here is relatively advanced, so trust or skip it if you don't have the needed linear algebra background. In math speak 'normal' sometimes means orthogonal. Here is the sense that 'normal' modes are orthogonal to each other. First, the matrix $[M]$ is generally both symmetric, non-singular and even positive definite, so $[M]$

- Has a symmetric inverse $[M]^{-1}$ with

$$[M]^{-1}[M] = [M][M]^{-1} = [\mathbf{1}] = [\text{identity matrix}],$$

- Has a unique positive definite square root $\sqrt{[M]}$ with

$$\sqrt{[M]}\sqrt{[M]} = [M],$$

- Has a unique inverse square root $[M]^{-1/2}$ with

$$[M]^{-1/2}[M][M]^{-1/2} = [\mathbf{1}].$$

First we use $[M]^{-1/2}$ to change coordinates from $\boldsymbol{x}$ to $\boldsymbol{y}$ as

$$\boldsymbol{x} = [M]^{-1/2}\boldsymbol{y} \quad \text{or} \quad \boldsymbol{y} = [M]^{1/2}\boldsymbol{x}.$$

Now we substitute this into the basic vibration equation ($[M]\ddot{\boldsymbol{x}} + [K]\boldsymbol{x} = \mathbf{0}$) and pre-multiply the whole equation by $[M]^{-1/2}$ to get

$$\underbrace{[M]^{-1/2}[M][M]^{-1/2}}_{[1]}\ddot{\boldsymbol{y}} + \underbrace{[M]^{-1/2}[K][M]^{-1/2}}_{[A]}\boldsymbol{y} = \mathbf{0}$$

$$\Rightarrow \quad \ddot{\boldsymbol{y}} + [A]\boldsymbol{y} = \mathbf{0}.$$

Now we look for solutions for $\boldsymbol{y}$ exactly as we did for $\boldsymbol{x}$ before. But, as opposed to $[B]$, $[A]$ is symmetric. So $[A]$ has n linear independent and mutually orthogonal *eigen vectors* $\boldsymbol{W}^1, \boldsymbol{W}^2, ...\boldsymbol{W}^n$ with associated *eigen values* $\lambda_1^2, \lambda_2^2, ..., \lambda_n^2$. These $\boldsymbol{W}_i$ give the $\boldsymbol{V}_i$ by $\boldsymbol{V}_i = [M]^{-1/2}\boldsymbol{W}_i$. The λ_i are the same.

These coordinate-changed $\boldsymbol{W}_i = [M]^{1/2}\boldsymbol{V}_i$ are 'normal' (mutually orthogonal) but the more physical $\boldsymbol{V}_i$ are not, even though the $\boldsymbol{V}_i$ are called 'normal' modes. Or you can say that the $\boldsymbol{V}_i$ are mutually orthogonal with respect to the weighting $[M]$, *e.g.*, $V_2'[M]V_5 = 0$.