Problems for Chapter 10

1D Vibrations

10.1 Free Vibration of a SDOF System

Preparatory Problems

10.1.1 A mass m is connected to a spring k and released from rest with the spring stretched a distance d from its static equilibrium position. It then oscillates back and forth repeatedly crossing the equilibrium. How much time passes from release until the mass moves through the equilibrium position for the second time? Neglect gravity and friction. Answer in terms of some or all of m, k, and d. *

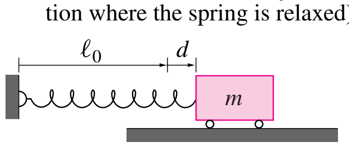
10.1.2 A spring k with rest length ℓ_0 is attached to a mass m which slides frictionlessly on a horizontal ground as shown. At time $t = 0$ the mass is released from rest with the spring stretched a distance d. Measure the mass position x relative to the wall.

a) What is the acceleration of the mass just after release?

b) Find a differential equation which describes the horizontal motion x of the mass.

c) What is the position of the mass at an arbitrary time t?

d) What is the speed of the mass when it passes through $x = \ell_0$ (the position where the spring is relaxed)?



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Problem 10.1.2

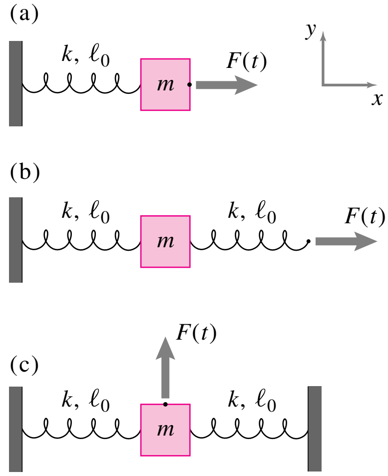
10.1.3 Reconsider the spring-mass system from problem 10.1.2.

a) Find the potential and kinetic energy of the spring mass system as functions of time.

b) Assigning numerical values to the various variables, use a computer to make a plot of the potential and kinetic energy as a function of time for several periods of oscillation. Are the potential and kinetic energy ever equal at the same time? If so, at what position $x(t)$?

c) Make a plot of kinetic energy versus potential energy. What is the phase relationship between the kinetic and potential energy?

10.1.4 For the three spring-mass systems shown in the figure, find the equation of motion of the mass in each case. All springs are massless and are shown in their relaxed states. Ignore gravity. (In problem (c) assume vertical motion.) *



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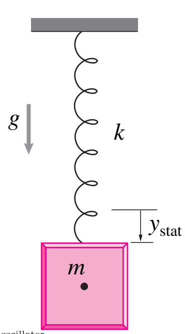
Problem 10.1.4

10.1.5 A mass-spring oscillator hangs vertically under gravity. The mass rests in static equilibrium by stretching the spring by an amount $y_{static} = 0.025\,\mathrm{m}$. Take your favorite value of g and find the natural frequency of the oscillator. How much time does the oscillator take to complete one oscillation?



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Problem 10.1.5

10.1.6 A mass moves on a frictionless surface. It is connected to a dashpot with damping coefficient b to its right and a spring with constant k and rest length ℓ to its left. At the instant of interest, the mass is moving to the right and the spring is stretched a distance x from its position where the spring is unstretched. There is gravity.

a) Draw a free body diagram of the mass at the instant of interest.

b) Derive the equation of motion of the mass. *

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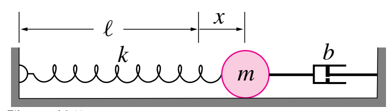

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Problem 10.1.6

10.1.7 A mass-spring-dashpot system has $m = 1\,\mathrm{kg}$, $k = 10\,\mathrm{kN/m}$, and $c = 5\,\mathrm{kg/\,s}$. Find the natural frequency, damping ratio, and the damped frequency of the system. Specify whether the system is underdamped, critically damped or overdamped.

10.1.8 The natural frequency, λ_n, of a SDOF system is 150 rad/s. Find the minimum damping (ξ) that the system must have for the resonant frequency to occur below 100 rad/s?

More-Involved Problems

10.1.9 The equation of motion of an unforced mass-spring-dashpot system is, $m\ddot{x} + c\dot{x} + kx = 0$, as discussed in the text. For a system with $m = 0.4\,\mathrm{kg}$, $c = 10\,\mathrm{kg/\,s}$, and $k = 5\,\mathrm{N/m}$,

a) Find whether the system is underdamped, critically damped, or overdamped.

b) Sketch a typical solution of the system.

c) Make an accurate plot of the response of the system (displacement vs time) for the initial conditions $x(0) = 0.1\,\mathrm{m}$ and $\dot{x}(0) = 0$.

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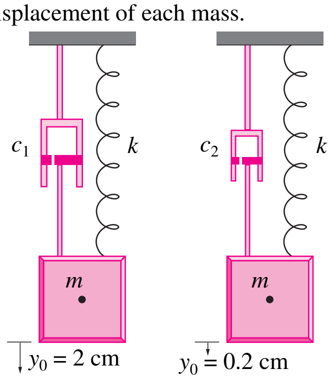
10.1.10 You are given to design a SDOF damped oscillator that should show no oscillations at all when disturbed from the equilibrium (*i.e.,* , it should return to equilibrium without overshooting on the other side). You are given a spring with stiffness

$k = 500\,\mathrm{N/m}$, a hydraulic damper with $c = 10\,\mathrm{kg/s}$, and you have a choice of masses from $m = 1\,\mathrm{kg}$ to $m = 10\,\mathrm{kg}$ in the increments of half kg. Find the appropriate mass.

10.1.11 Two SDOF oscillators with the same k and m but different c's are hung from the ceiling as shown in the figure. The one on the left is pulled down 2 cm and let go. The other is pulled down by 0.2 cm and let go. Which oscillator undergoes more number of oscillations before reaching the steady state. Find the steady state displacement of each mass.



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Problem 10.1.11

10.1.12 Experiments conducted on free oscillations of a damped oscillator reveal that the amplitude of oscillations drops to 25% of its peak value in just 3 periods of oscillations. The period os oscillation is measured to be 0.6 s and the mass of the system is known to be 1.2 kg. Find the damping coefficient and the spring stiffness of the system.

10.1.13 You are required to design a mass-spring-dashpot system that, if disturbed, returns to its equilibrium position the quickest. You are given a mass, $m = 1\,\mathrm{kg}$, and a damper with $c = 10\,\mathrm{kg/s}$. What should be the stiffness of the spring? Your solution needs to include your definition of "'quickest".

10.2 Forced Vibration and resonance

Preparatory Problems

10.2.1 Given that $\ddot{\theta} + k^2\theta = \beta \sin \omega t$, $\theta(0) = 0$, and $\dot{\theta}(0) = \dot{\theta}_0$, find $\theta(t)$.

10.2.2 Three SDOF systems, each with the same mass but different stiffnesses, $k_1 = k$, $k_2 = 2k$, and $k_3 = 4k$, and different damping, $c_1 = c$, $c_2 = 2c$, and $c_3 = c/2$, are subjected to the same periodic forcing, $F = F_0 \sin pt$ where p is less than the resonant frequency of each of the systems. Sketch approximately the response of each of the three oscillators assuming all of them to be underdamped. Clearly mark the transient and steady state part of the response, and indicate the relative values of the response amplitudes.

10.2.3 A 3 kg mass is suspended by a spring ($k = 10\,\mathrm{N/m}$) and forced by a 5 N sinusoidally oscillating force with a period of 1 s. What is the amplitude of the steady-state oscillations (ignore the "homogeneous" solution)

More-Involved Problems

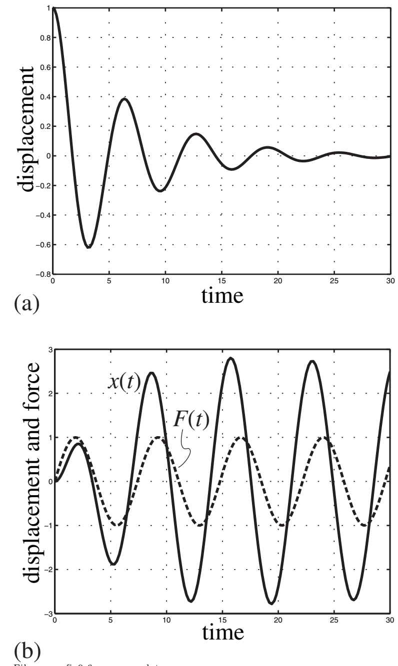
10.2.4 A machine that can be modeled as a SDOF system is put under vibration test for estimating the system parameters m, k, and c. First, a transient test is conducted by disturbing the machine from its equilibrium and letting its settle down to equilibrium again. The transient response is recorded as a displacement versus time plot and is shown in fig. 10.2.4(a). Next, a sinusoidal forcing is of amplitude F_0 and angular frequency p is applied on the machine and its steady state response is recoded along with the forcing function. This response is shown in fig. 10.2.4(b).

a) Mark the relevant points on the transient response plot and explain, with equations, which systems parameters can be determined using what information from this plot.

b) On the steady state plot, mark the phase difference between the response and the forcing function. From the given phase, can you find out whether $p > \lambda_n$ or $p < \lambda_n$?

c) From the phase difference of the steady state response and the information obtained from the transient response, can you determine the frequency ratio r? Explain with appropriate equations.

d) From the amplitude of the steady state response, and the rest of the information obtained above, find the rest of the system parameters.



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Problem 10.2.4

10.2.5 A machine produces a steady-state vibration due to a forcing function described by $Q(t) = Q_0 \sin \omega t$, where $Q_0 = 5000N$. The machine rests on a circular concrete foundation. The foundation rests on an isotropic, elastic half-space. The equivalent spring constant of the half-space is $k = 2,000,000\,\mathrm{N{\cdot}m}$ and has a damping ratio $d = c/c_c = 0.125$. The machine operates at a frequency of $\omega = 4$ Hz.

1. What is the natural frequency of the system?
2. If the system were undamped, what would the steady-state displacement be?
3. What is the steady-state displacement given that $d = 0.125$?
4. How much additional thickness of concrete should be added to the footing to reduce the damped steady-state amplitude by 50%? (The diameter must be held constant.)

10.2.6 The transient response of an oscillatory system shows exponential decay of the peak displacements at each cycle. The second peak is found to be twice as big as the fifth peak. Find the damping ratio xi for the system. How many cycles does it take for the peak displacement to drop below 5% of the first peak displacement?

10.2.7 A 50 kg engine is mounted on springs with an equivalent single spring stiffness of 1200 N/m. Using various means, enough damping needs to be provided so that any unwanted vibration dies quickly. Assume that this objective is met by dissipating 80% of the available energy in a single cycle of vibration. Find the damping coefficient of the system.

10.2.8 Consider the system shown in the figure. You are given that $m = 10\,\mathrm{kg}$, $k = 50\,\mathrm{N/m}$, and $c = 5\,\mathrm{kg/s}$. A periodic force $F = F_0 \cos pt$ acts on the system as shown where $F_0 = 25\,\mathrm{N}$ and $p = 2.5\,\mathrm{rad/s}$.

a) Find the resonant frequency of the system.

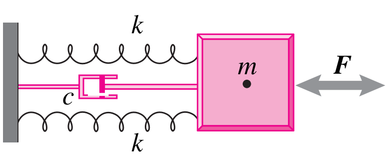
b) Find the steady state response of the system, specifying the amplitude and phase of the motion.

c) What is the displacement amplification ($G = A/(F_0/k)$)?

d) Find the work done by the force on the system in one cycle.

e) Find the energy lost to the damper in one cycle.

f) Find the quality factor, Q, of the system using the energy calculations.



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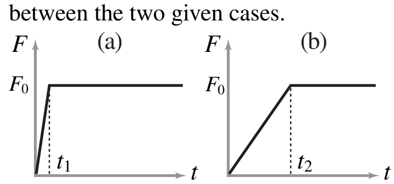
Problem 10.2.8

10.2.9 A MEMS cantilever beam resonator is used for mass measurement of biological molecules by comparing the shift in the resonant frequency of the beam after the test molecule is attached to the free end of the beam. In a SDOF model of the resonator, it is equivalent to finding the difference in the resonant frequency of the system with mass m and $m + \Delta m$. If the 'effective mass' of the beam (mass to be used in the SDOF model) is 2.05×10^{-15} kg, the stiffness is $0.625\,\mathrm{N/m}$, and the Q of the resonator is 900, find the shift in the resonant peak in Hz when a biological molecule of mass 1.36×10^{-21} kg is attached to end of the beam (equivalently to the mass m).

10.2.10 A damped mass-spring system is subjected to a constant load $F_0 = 50\,\mathrm{N}$ by ramping the load to the constant level in (a) $t_1 = 2\,\mathrm{s}$ and (b) $t_2 = 10\,\mathrm{s}$. If the mass of the system $m = 1\,\mathrm{kg}$, the natural frequency $\lambda_n = 62\,\mathrm{rad/s}$, and the damping ratio $\xi = 0.2$, find the difference in the settling time of the system to the steady state between the two given cases.



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Problem 10.2.10

10.2.11 **An accelerometer** is a sensor that is used to measure acceleration of a body. It can be modeled as a single degree of freedom spring-mass-dashpot system that is attached to a body frame as shown in the figure. Assume that the body undergoes vertical motion denoted by $y(t)$ and, as a result, the mass of the accelerometer undergoes vertical motion $z(t)$ relative to the frame. From a measurement of $z(t)$ we want to know if we can determine the acceleration $\ddot{y}$ of the frame. Neglect gravity.

a) What is the absolute or inertial acceleration of the mass in terms of $z(t)$ and $y(t)$ and their derivatives?

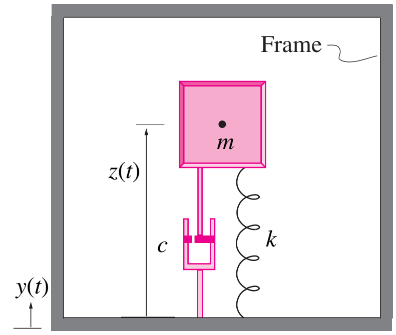
b) Write the equation of motion of the mass in terms of z and y.

c) Assume that $y(t) = y_0 \sin pt$. What is the magnitude of acceleration of the frame (that is, the peak acceleration of this sine wave)?

d) Find the steady state response $z(t)$ of the accelerometer when $y(t) = y_0 \sin pt$ (a big mess). Plot the amplitude of the response vs the amplitude of the frame acceleration as the frequency p is varied.

e) One would like a signal (z) that is proportional to acceleration $\ddot{y}$ independent of the frequency of shaking. Show that this requires that the frequency of frame motion p must be much smaller than the natural frequency λ_n.

f) What is the amplitude of the response (the magnitude of z) of the accelerometer when $p \gg \lambda_n$?



Filename:pfig9-6-accelerometer
Problem 10.2.11: A single degree of freedom spring-mass model of an accelerometer.

10.2.12 Consider the accelerometer described in Problem 10.2.11. Assume that the frame undergoes a sinusoidal motion given by $y(t) = y_0 \sin pt$.

a) Find the response $z(t)$ of the accelerometer.

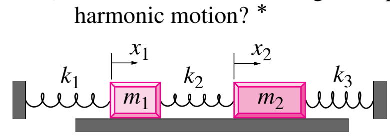
b) Given that $m = 0.5\,\mathrm{kg}$, $k = 5\,\mathrm{kN/m}$, and $c = 10\,\mathrm{kg/s}$, find the maximum acceleration that the accelerometer can sense, assuming the accelerometer to work in the frequency range much below its natural frequency (*i.e.*, $p/\lambda_n \ll 1$). Express your answer in terms of the gravitational acceleration g (it is customary to talk about acceleration of various things in terms of 'so many g's').

10.3 Normal Modes

10.3.1 A two degree of freedom mass-spring system, made up of two unequal masses m_1 and m_2 and three springs with unequal stiffnesses k_1, k_2 and k_3, is shown in the figure. All three springs are relaxed in the configuration shown. Neglect friction.

a) Derive the equations of motion for the two masses.

b) Does each mass undergo simple harmonic motion? *



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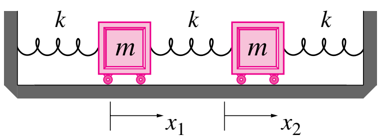
Problem 10.3.1

10.3.2 Normal Modes. Three equal springs (k) hold two equal masses (m) in place. There is no friction. x_1 and x_2 are the displacements of the masses from their equilibrium positions.

a) How many independent normal modes of vibration are there for this system? *

b) Assume the system is in a normal mode of vibration and it is observed that $x_1 = A\sin(ct) + B\cos(ct)$ where A, B, and c are constants. What is $x_2(t)$? (The answer is not unique. You may express your answer in terms of any of A, B, c, m and k.) *

c) Find all of the frequencies of normal-mode-vibration for this system in terms of m and k. *



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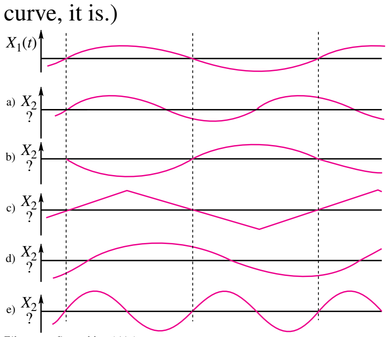
Problem 10.3.2

10.3.3 $x_1(t)$ and $x_2(t)$ are measured positions on two points of a vibrating structure. $x_1(t)$ is shown. Some candidates for $x_2(t)$ are shown. Which of the $x_2(t)$ could possibly be associated with a normal mode vibration of the structure? Answer "could" or "could not" next to each choice and briefly explain your answer (If a curve looks like it is meant to be a sine/cosine curve, it is.)



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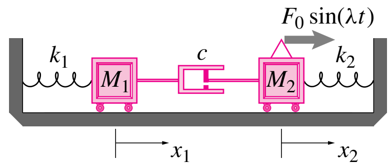
Problem 10.3.3

More-Involved Problems

10.3.4 Two masses are connected to fixed supports and each other with the two springs and dashpot shown. The displacements x_1 and x_2 are defined so that $x_1 = x_2 = 0$ when both springs are unstretched.

For the special case that $C = 0$ and $F_0 = 0$ clearly define two different set of initial conditions that lead to normal mode vibrations of this system.



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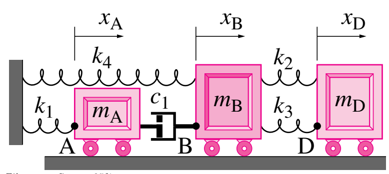
Problem 10.3.4

10.3.5 As in problem 9.4.11, a system of three masses, four springs, and one damper are connected as shown. Assume that all the springs are relaxed when $x_A = x_B = x_D = 0$.

a) In the special case when $k_1 = k_2 = k_3 = k_4 = k$, $c_1 = 0$, and $m_A = m_B = m_D = m$, find a normal mode of vibration. Define it in any clear way and explain or show why it is a normal mode in any clear way. *

b) In the same special case as in (a) above, find another normal mode of vibration. *



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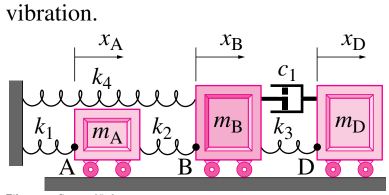
Problem 10.3.5

10.3.6 As in problem 9.4.10, a system of three masses, four springs, and one damper are connected as shown. In the special case when $c_1 = 0$, find the normal modes of vibration.



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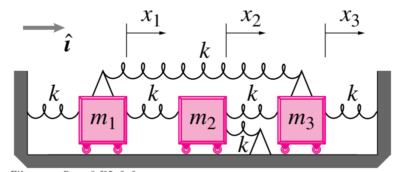
Problem 10.3.6

10.3.7 Normal modes. All three masses have $m = 1\,\text{kg}$ and all 6 springs are $k = 1\,\text{N/m}$. The system is at rest when $x_1 = x_2 = x_3 = 0$.

a) Find as many different initial conditions as you can for which normal mode vibrations result. In each case, find the associated natural frequency. (we will call two initial conditions $[v]$ and $[w]$ different if there is no constant c so that $[v_1\ v_2\ v_3] = c[w_1\ w_2\ w_3]$. Assume the initial velocities are zero.)

b) For the initial condition $[x_0] = [\,0.1\,\text{m}\ \ 0\ \ 0\,]$, $[\dot{x}_0] = [\,0\ \ 2\,\text{m/s}\ \ 0\,]$ what is the initial (immediately after the start) acceleration of mass 2?



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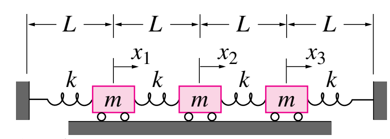
Problem 10.3.7

10.3.8 For the three-mass system shown, assume $x_1 = x_2 = x_3 = 0$ when all the springs are fully relaxed. One of the normal modes is described with the initial condition $(x_{1_0}, x_2, x_3) = (1, 0, -1)$.

a) What is the angular frequency ω for this mode? Answer in terms of L, m, k, and g. (Hint: Note that in this mode of vibration the middle mass does not move.) *

b) Make a neat plot of x_2 versus x_1 for one cycle of vibration with this mode.



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Problem 10.3.8