SAMPLE 10.1 **Springs in series versus springs in parallel:** Two massless springs with spring constants k_1 and k_2 are attached to mass A *in parallel* (although they look superficially as if they are in series) as shown in Fig. 10.5. An identical pair of springs is attached to mass B *in series*. Taking $m_A = m_B = m$, find and compare the natural frequencies of the two systems. Ignore gravity.

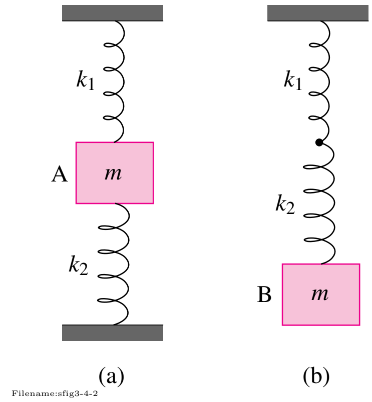

Figure 10.4:

Solution Let us pull each mass downwards by a small vertical distance y and then release. Measuring y to be positive downwards, we can derive the equations of motion for each mass by writing the balance of linear momentum for each as follows.

- **Mass A:** The free body diagram of mass A is shown in Fig. 10.6. As the mass is displaced downwards by y, spring 1 gets stretched by y whereas spring 2 gets compressed by y. Therefore, the forces applied by the two springs, $k_1 y$ and $k_2 y$, are in the same direction. The linear momentum balance of mass A in the vertical direction gives:

$$\begin{aligned} \sum F &= m a_y \\ \text{or} \quad -k_1 y - k_2 y &= m\ddot{y} \\ \text{or} \quad \ddot{y} + \left(\frac{k_1 + k_2}{m}\right) y &= 0. \end{aligned}$$

Let the natural frequency of this system be ω_p. Comparing with the standard simple harmonic equation $\ddot{x} + \lambda^2 x = 0$ (see box C.1 on page 1016), we get the natural frequency (λ) of the system:

$$\omega_p = \sqrt{\frac{k_1 + k_2}{m}} \tag{10.11}$$

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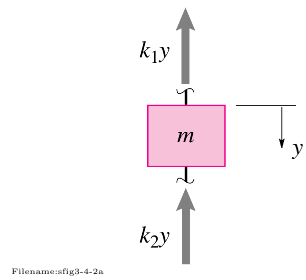

Figure 10.5: Free body diagram of the mass.

- **Mass B:** The free body diagram of mass B and the two springs is shown in Fig. 10.7. In this case both springs stretch as the mass is displaced downwards. Let the net stretch in spring 1 be y_1 and in spring 2 be y_2. y_1 and y_2 are unknown, of course, but we know that

$$y_1 + y_2 = y \tag{10.12}$$

Now, using the free body diagram of spring 2 and then writing linear momentum balance we get,

$$\begin{aligned} k_2 y_2 - k_1 y_1 &= \underbrace{m}_{0} \quad a = 0 \\ y_1 &= \frac{k_2}{k_1} y_2 \end{aligned} \tag{10.13}$$

Solving (10.12) and (10.13) we get

$$y_2 = \frac{k_1}{k_1 + k_2} y.$$

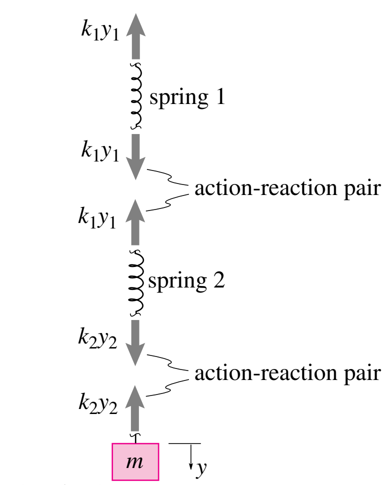

Figure 10.6: Free body diagrams

Now, linear momentum balance of mass B in the vertical direction gives:

$$-k_2 y_2 = m a_y = m\ddot{y}$$

$$\text{or} \quad m\ddot{y} + k_2 \overbrace{\frac{k_1}{k_1+k_2} y}^{y_2} = 0$$

$$\text{or} \quad \ddot{y} + \frac{k_1 k_2}{m(k_1+k_2)} y = 0. \tag{10.14}$$

Let the natural frequency of this system be denoted by ω_s. Then, comparing with the standard simple harmonic equation as in the previous case, we get

$$\omega_s = \sqrt{\frac{k_1 k_2}{m(k_1+k_2)}}. \tag{10.15}$$

$$\omega_s = \sqrt{\frac{k_1 k_2}{m(k_1+k_2)}}$$

From (10.11) and (10.15)

$$\frac{\omega_p}{\omega_s} = \frac{k_1+k_2}{\sqrt{k_1 k_2}}.$$

Let $k_1 = k_2 = k$. Then, $\omega_p/\omega_s = 2$, *i.e.,* the natural frequency of the system with two identical springs in parallel is twice as much as that of the system with the same springs in series. Intuitively, the restoring force applied by two springs in parallel will be more than the force applied by identical springs in series. In one case the restoring forces add and in the other they don't. Therefore, we do expect mass A to oscillate at a faster rate (higher natural frequency) than mass B. The ratio of the two natural frequencies, ω_p/ω_s as a function of k_1/k_2 is plotted in fig. 10.8. As we can see, ω_p is always higher than ω_s for all k_1/k_2 ratios and its minimum value is twice that of ω_s when $k_1 = k_2$.

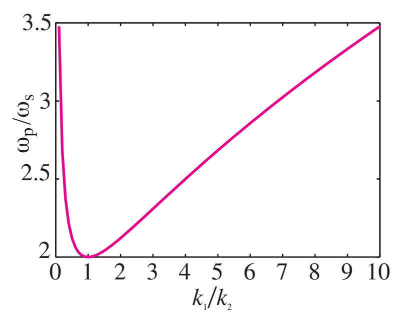

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Figure 10.7: The ratio ω_p/ω_s of the two natural frequencies as a function of the spring stiffness ratio k_1/k_2.

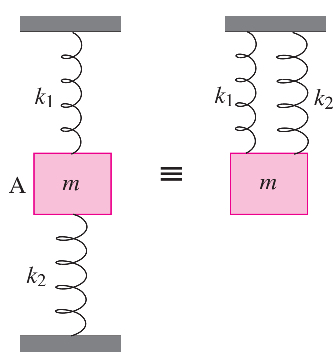

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Figure 10.8: The two arrangements of springs are equivalent; they are connected in parallel.

Comments:

1. Although the springs attached to mass A do not visually seem to be in parallel, from mechanics point of view they are parallel (see fig. 10.9). You can easily check this result by putting the two springs visually in parallel and then deriving the equation of mass A. You will get the same equations. For springs in parallel, each spring has the *same displacement* but *different forces*. For springs in series, each has *different displacements* (if $k_1 \neq k_2$) but the *same force*.
2. When many springs are connected to a mass in series or in parallel, sometimes we talk about their *effective* spring constant, *i.e.,* the spring constant of a single imaginary spring which could be used to replace all the springs attached in parallel or in series. Let the effective spring constant for springs in parallel and in series be represented by k_{pe} and k_{se} respectively. By comparing eqns. (10.11) and (10.15) with the expression for natural frequency of a simple spring mass system, we see that

   $$k_{pe} = k_1 + k_2 \quad \text{and} \quad \frac{1}{k_{se}} = \frac{1}{k_1} + \frac{1}{k_2}.$$

   These expressions can be easily extended for any arbitrary number of springs, say, N springs:

   $$k_{pe} = k_1 + k_2 + \ldots + k_N \quad \text{and} \quad \frac{1}{k_{se}} = \frac{1}{k_1} + \frac{1}{k_2} + \ldots + \frac{1}{k_N}.$$