SAMPLE 10.2 Figure 10.10 shows two responses obtained from experiments on two spring-mass systems. For each system

1. Find the natural frequency.
2. Find the initial conditions.

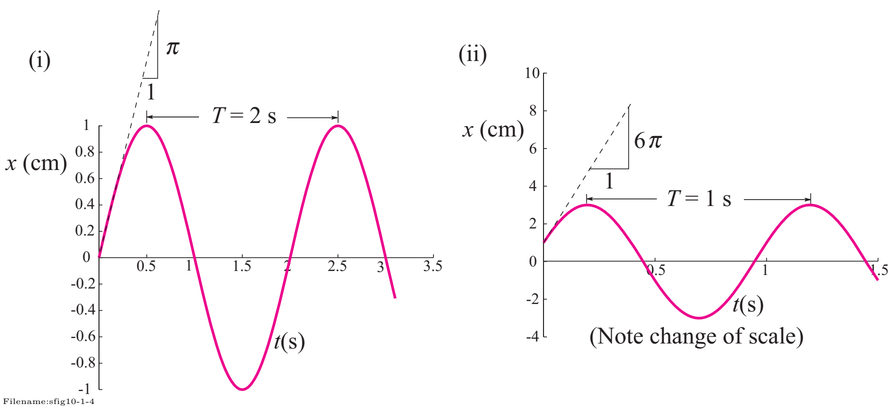

Figure 10.9:

Solution

1. **Natural frequency:** By definition, the natural frequency f is the number of cycles the system completes in one second. From the given responses we see that:

   Case(i): the system completes $\frac{1}{2}$ a cycle in 1 s.

$$\Rightarrow \qquad f = \frac{1}{2}\text{ Hz}.$$

   Case(ii): the system completes 1 cycle in 1 s.

$$\Rightarrow \qquad f = 1\text{ Hz}.$$

   It is usually hard to measure the fraction of cycle occurring in a short time. It is easier to first find the time period, *i.e.,* the time taken to complete 1 cycle. ② Then the natural frequency can be found by the formula $f = \frac{1}{T}$. From the given responses, we find the time period by estimating the time between two successive peaks (or troughs): From Figure 10.10 we find that for

② To estimate the frequency of some repeated motion in an experiment, it is best to measure the time for a large number of cycles, say 5, 10 or 20, and then divide that time by the total number of cycles to get an average value for the time period of oscillation.

   Case (i):

$$f = \frac{1}{T} = \frac{1}{2\,\text{s}} = \frac{1}{2}\text{ Hz},$$

   Case (ii):

$$f = \frac{1}{T} = \frac{1}{1\,\text{s}} = 1\text{ Hz}$$

$$\text{case (i) } f = \tfrac{1}{2}\text{ Hz.} \quad \text{case (ii) } f = 1\text{ Hz.}$$

2. **Initial conditions:** Now we are to find the displacement and velocity at $t = 0$ s for each case. Displacement is easy because we are given the displacement plot, so we just read the value at $t = 0$ from the plots:

   Case (i):

$$x(0) = 0.$$

   Case (ii):

$$x(0) = 1\text{ cm}.$$

The velocity (actually the speed) is the time-derivative of the displacement. Therefore, we get the initial velocity from the slope of the displacement curve at $t = 0$.

Case (i):

$$\dot{x}(0) = \frac{dx}{dt}(t=0) = \frac{\pi \text{ cm}}{1 \text{ s}} = 3.14 \text{ cm/s}.$$

Case (ii):

$$\dot{x}(0) = \frac{dx}{dt}(t=0) = \frac{6\pi \text{ cm}}{1 \text{ s}} = 18.85 \text{ cm/s}.$$

Thus the initial conditions are

Case (i):	$x(0) = 0$	$\dot{x}(0) = 3.14 \text{ cm/s}$
Case (ii):	$x(0) = 1 \text{ cm}$	$\dot{x}(0) = 18.85 \text{ cm/s}$

Comments: Estimating the speed from the initial slope of the displacement curve at $t = 0$ is not a very good method because it is hard to draw an accurate tangent to the curve at $t = 0$. A slightly different line but still seemingly tangential to the curve at $t = 0$ can lead to significant error in the estimated value. A better method, perhaps, is to use the known values of displacement at different points and use the energy method to calculate the initial speed. We show sample calculations for the first system:

Case(i): We know that $x(0) = 0$. Therefore the entire energy at $t = 0$ is the kinetic energy $= \frac{1}{2}mv_0^2$. At $t = 0.5$ s we note that the displacement is maximum, *i.e.,* the speed is zero. Therefore, the entire energy is potential energy $= \frac{1}{2}kx^2$, where $x = x(t = 0.5\text{ s}) = 1$ cm.

Now, from the conservation of energy:

$$\begin{aligned}
\frac{1}{2}mv_0^2 &= \frac{1}{2}k(x_{t=0.5\text{ s}})^2 \\
\Rightarrow \quad v_0 &= \sqrt{\frac{k}{m}} \cdot (x_{t=0.5\text{ s}}) \\
&= \underbrace{\sqrt{\frac{k}{m}}}_{\lambda} \cdot (1 \text{ cm}) \\
&= 2\pi f \cdot (1 \text{ cm}) \\
&= 2\pi \cdot \frac{1}{2} \text{ Hz} \cdot 1 \text{ cm} \\
&= 3.14 \text{ cm/s}.
\end{aligned}$$

Similar calculations can be done for the second system.