SAMPLE 10.4 A structure, modeled as a single degree of freedom system, exhibits characteristics of an underdamped system under free oscillations. The response of the structure to some initial condition is determined to be $x(t) = Ae^{-\xi\lambda t}\sin(\lambda_D t)$ where $A = 0.3\,\mathrm{m}$, $\xi \equiv$ damping ratio $= 0.02$, $\lambda \equiv$ undamped circular frequency $= 1\,\mathrm{rad/s}$, and $\lambda_D \equiv$ damped circular frequency $= \lambda\sqrt{1-\xi^2} \approx \lambda$.

1. Find an expression for the ratio of energies of the system at the $(n+1)$th displacement peak and the nth displacement peak.
2. What percent of energy available at the first peak is lost after 5 cycles?

Solution

1. We are given that

$$x(t) = Ae^{-\xi\lambda t}\sin(\lambda_D t).$$

The structure attains its first displacement peak when $\sin\lambda_D t$ is maximum, *i.e.,*

$$\lambda_D t = \frac{\pi}{2} \quad \Rightarrow \quad t = \frac{\pi}{2\lambda_D}.$$

At this instant,

$$\begin{aligned} x(t) &= Ae^{-\xi\cdot\lambda\cdot\frac{\pi}{2\lambda_D}} \\ &= Ae^{-\frac{\pi}{2}\cdot\frac{\xi}{\sqrt{1-\xi^2}}} \\ &= (0.3\,\mathrm{m})\cdot e^{-0.0314} \\ &= 0.29\,\mathrm{m}. \end{aligned}$$

Let x_n and x_{n+1} be the values of the displacement at the nth and the $(n+1)$th peak, respectively. Since x_n and x_{n+1} are peak displacements, the respective velocities are zero at these points. Therefore, the energy of the system at these peaks is given by the potential energy stored in the spring. That is

$$E_n = \frac{1}{2}kx_n^2 \quad \text{and} \quad E_{n+1} = \frac{1}{2}kx_{n+1}^2. \tag{10.18}$$

Let t_n be the time at which the nth peak displacement x_n is attained, *i.e.,*

$$x_n = Ae^{-\xi\lambda t_n} \tag{10.19}$$

Since x_{n+1} is the next peak displacement, it must occur at $t = t_n + T_D$ where T_D is the time period of damped oscillations. Thus

$$x_{n+1} = Ae^{-\xi\lambda(t_n+T_D)} \tag{10.20}$$

From Eqns (10.18), (10.19), and (10.20)

$$\begin{aligned} \frac{E_{n+1}}{E_n} &= \frac{\frac{1}{2}k(Ae^{-\xi\lambda(t_n+T_D})^2}{\frac{1}{2}k(Ae^{-\xi\lambda t_n})^2} \\ &= e^{-2\xi\lambda T_D}. \end{aligned}$$

$$\frac{E_{n+1}}{E_n} = e^{-2\xi\lambda T_D}.$$

2. Noting that $T_D = \frac{2\pi}{\lambda_D}$ and $\lambda_D = \lambda\sqrt{1-\xi^2}$, we get

$$\begin{aligned} E_{n+1} &= E_n\, e^{-2\xi\cancel{\lambda}\cdot\frac{2\pi}{\cancel{\lambda}\sqrt{1-\xi^2}}} \\ &= e^{-4\pi\frac{\xi}{\sqrt{1-\xi^2}}} \approx e^{-4\pi\xi} \\ \Rightarrow \quad E_{n+1} &= e^{-4\pi\xi}\, E_n. \end{aligned}$$

Applying this equation recursively for $n = n-1, n-2, \ldots, 1, 0,$, we get

$$\begin{aligned} E_n &= e^{-4\pi\xi}.E_{n-1} \\ &= e^{-4\pi\xi}.(e^{-4\pi\xi}\cdot E_{n-2}) \\ &= (e^{-4\pi\xi})^3.E_{n-3} \\ &\vdots \\ &= (e^{-4\pi\xi})^n.E_0. \end{aligned}$$

Now we use this equation to find the percentage of energy of the first peak ($n = 0$) lost after 5 cycles ($n = 5$):

$$\begin{aligned} \Delta E_5 &= \frac{E_0 - E_5}{E_0} \times 100 \\ &= \left(1 - e^{-4\pi\xi\cdot 5}\right) \times 100 \\ &= 71.5\%. \end{aligned}$$

$$\Delta E_5 = 71.5\%.$$