SAMPLE 10.7 Response to a constant force: A constant force $F = 50\,\text{N}$ acts on a mass-spring system as shown in the figure. Let $m = 5\,\text{kg}$ and $k = 10\,\text{kN/m}$.

1. Write the equation of motion of the system.
2. If the system starts from the initial displacement $x_0 = 0.01$ m with zero velocity, find the displacement of the mass as a function of time.
3. Plot the response (displacement) of the system against time and describe how it is different from the unforced response of the system.

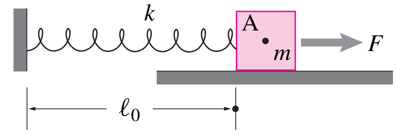

Figure 10.20:

Solution

1. The free-body diagram of the mass is shown in fig. 10.22 at a displacement x (assumed positive to the right). Applying linear momentum balance in the x-direction, *i.e.*, $(\sum \vec{F} = m\vec{a}) \cdot \hat{\imath}$, we get

$$\begin{aligned} F - kx &= m\ddot{x} \\ \Rightarrow \quad m\ddot{x} + kx &= F \end{aligned} \tag{10.38}$$

which is the equation of motion of the system.

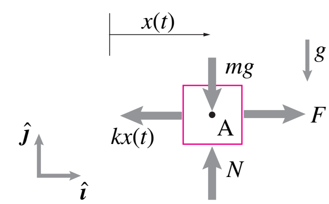

Figure 10.21: Free-body diagram of the mass.

2. The equation of motion has a non-zero right hand side. Thus, it is a nonhomogeneous differential equation. A general solution of this equation is made up of two parts — the homogeneous solution x_h which is the solution of the unforced system (eqn. (10.38) with $F = 0$), and a particular solution x_p that satisfies the nonhomogeneous equation. Thus,

$$x(t) = x_h(t) + x_p(t). \tag{10.39}$$

Now, let us find $x_h(t)$ and $x_p(t)$.

Homogeneous solution: $x_h(t)$ has to satisfy the homogeneous equation

$$m\ddot{x} + kx = 0.$$

Let $\lambda = \sqrt{k/m}$. Then, from the solution of unforced harmonic oscillator, we know that

$$x_h(t) = A\sin(\lambda t) + B\cos(\lambda t)$$

where A and B are constants to be determined later from initial conditions.

Particular solution: x_p must satisfy eqn. (10.38). Since the nonhomogeneous part of the equation is a constant (F), we guess that x_p must be a constant too (of the same form as F). Let $x_p = C$. Now we substitute $x_p = C$ in eqn. (10.38) and solve the resulting equation to determine C:

$$m\underbrace{\ddot{C}}_{0} + kC = F \quad \Rightarrow \quad C = F/k \quad \text{or} \quad x_p = F/k.$$

Substituting x_h and x_p in eqn. (10.39), we get

$$x(t) = A\sin(\lambda t) + B\cos(\lambda t) + F/k. \tag{10.40}$$

Now we use the given initial conditions to determine A and B.

$$\begin{aligned} x(t=0) &= B + F/k = x_0 \text{ (given)} \quad \Rightarrow \quad B = x_0 - F/k \\ \dot{x}(t) &= A\lambda\cos(\lambda t) - B\lambda\sin(\lambda t) \\ \Rightarrow \quad \dot{x}(t=0) &= A = 0 \text{ (given)} \quad \Rightarrow \quad A = 0. \end{aligned}$$

Thus,

$$x(t) = (x_0 - F/k)\cos(\lambda t) + F/k, \tag{10.41}$$

$$\text{and} \quad \dot{x}(t) = -\lambda(x_0 - F/k)\sin(\lambda t). \tag{10.42}$$

3. Let us plug the given numerical values, $k = 10\,\mathrm{kN/m}, m = 5\,\mathrm{kg}$, (which gives $\lambda = \sqrt{k/m} = 44.72\,\mathrm{rad/s}$), $F = 50\,\mathrm{N}$ and $x_0 = 0.01\,\mathrm{m}$ in eqn. (10.41) and (10.42). The displacement and the velocity are now given as

$$\begin{aligned} x(t) &= (0.005\,\mathrm{m})\cos(44.72\,\mathrm{rad/s}\cdot t) + 0.005\,\mathrm{m}, \\ \text{and} \quad \dot{x}(t) &= -(0.22\,\mathrm{m/s})\sin(44.72\,\mathrm{rad/s}\cdot t). \end{aligned}$$

This response is plotted in fig. 10.23 against time. Note that the oscillations of the mass are about a non-zero mean value, $x_{\mathrm{eq}} = 0.005\,\mathrm{m}$. A little thought should reveal that this is what we should expect. When a mass hangs from a spring under gravity, the spring elongates a little, by mg/k to be precise, to balance the mass. Thus, the new static equilibrium position is not at the relaxed length ℓ_0 of the spring but at $\ell_0 + mg/k$. Any oscillations of the mass will be about this new equilibrium. The velocity, however, has a zero mean value which is what we expect from eqn. (10.42).

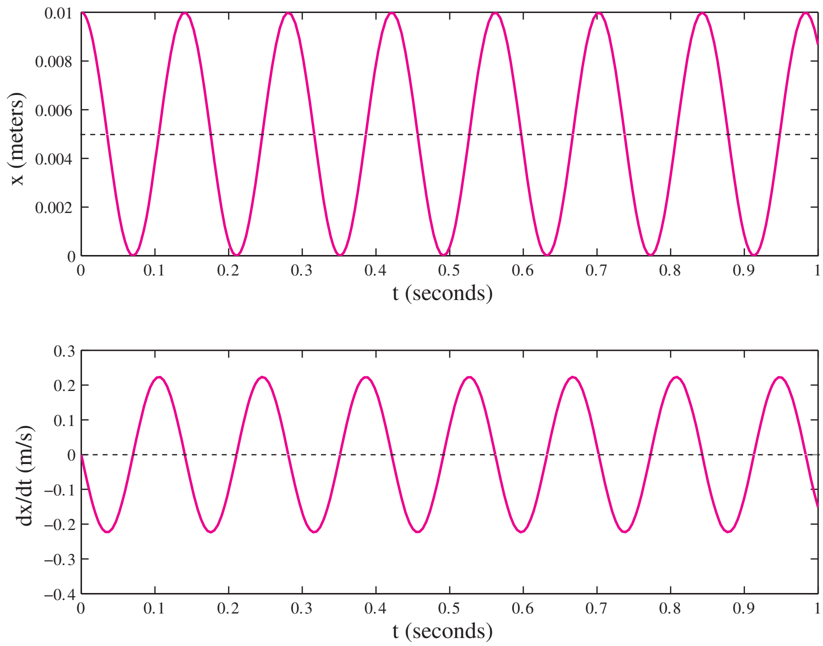

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Figure 10.22: Displacement of the mass as a function of time. Note that the mass oscillates about a nonzero value of x.

This problem is exactly like a mass hanging from a spring under gravity, a constant force, but just rotated by 90°. The new static equilibrium is at $x_{\mathrm{eq}} = F/k$ and any oscillations of the mass have to be around this new equilibrium.

We can rewrite the response of the system by measuring the displacement of the mass from the new equilibrium. Let $\tilde{x} = x - F/k$. Then, eqn. (10.41) becomes

$$\tilde{x} = \tilde{x}_0 \cos(\lambda t)$$

where $\tilde{x}_0 = x_0 - F/k$ is the initial displacement. Clearly, this is the response of an unforced harmonic oscillator. Thus the effect of a constant force on a spring-mass system is just a shift in its static equilibrium position.