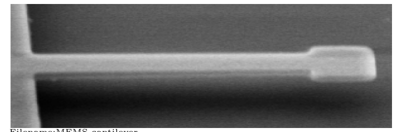
SAMPLE 10.10 A MEMS (microelectromechanical system) cantilever resonator (shown in the figure) is modeled as a single degree of freedom oscillator (SDOF) oscillator. Using load deflection measurements, the stiffness of the beam (equivalent to the spring stiffness) is found to be 90 N/m. The beam is excited using electrical actuation and its resonant frequency is determined under two different conditions: (i) the beam vibrating in vacuum where the viscous damping is negligible, and (ii) the beam vibrating in ambient conditions where the airflow around it causes viscous damping. If the two frequencies are found to be 30 kHz and 28.4 kHz, respectively, find the equivalent mass m and the damping ratio for the SDOF model. If the beam is subjected to a periodic actuation at the free end by a force $F(t) = F\sin(2\pi f t)$ where $F = 50\mu$ N and $f = 25$ kHz, find the steady state displacement amplitude and the phase of the free end of the resonator.



Filename:MEMS-cantilever

Figure 10.26: A MEMS cantilever resonator: Such resonators are typically silicon cantilever beams fabricated using micromachining process. These beams have geometric dimensions in micrometers and can be as small as a couple of micrometers long and a few nanometers in width and thickness.

Solution First we need to find m and c for the equivalent mass-spring-dashpot model. In the first case, where the resonant frequency is found in vacuum, we neglect damping, *i.e.,* $c = 0$. Therefore, the given frequency is the natural frequency. However, it is f_n, not the circular natural frequency λ_n. Now, $\lambda_n = 2\pi f_n$, hence

$$\sqrt{\frac{k}{m}} = 2\pi f_n \quad \Rightarrow \quad m = \frac{k}{4\pi^2 f_n^2} = \frac{90\,\mathrm{N/m}}{4\pi^2(30000\,\mathrm{s}^{-1})^2} = 2.533 \times 10^{-9}\,\mathrm{kg}.$$

We now use the damped natural frequency to find the damping ratio ξ. Since, we are given $f_d = 28.4$ kHz, and we know that $\lambda_d = \lambda_n\sqrt{(1-\xi^2)}$, we have

$$\begin{aligned} 2\pi f_d &= 2\pi f_n\sqrt{1-\xi^2} \\ \xi &= \sqrt{1-\left(\frac{f_d}{f_n}\right)^2} = \sqrt{1-\left(\frac{28.4\,\mathrm{kHz}}{30\,\mathrm{kHz}}\right)^2} \\ &= 0.32. \end{aligned}$$

Now, we know the values of all system parameters for our SDOF model of the MEMS resonator — m, k and ξ (can find c if required from ξ, k and m). For the given sinusoidal forcing, the equation of motion of the SDOF oscillator is:

$$m\ddot{x} + c\dot{x} + kx = F\sin(2\pi f t).$$

We can write the steady state solution as the particular solution $x(t) = A_0\sin(pt - \phi)$ where $p = 2\pi f$, and the displacement amplitude A_0 and the phase ϕ are given by the following expressions:

$$A_0 = \frac{F/k}{\sqrt{(2\xi r)^2 + (1-r^2)^2}}, \quad \text{and} \quad \phi = \tan^{-1}\left(\frac{2\xi r}{1-r^2}\right).$$

Since, $r = \frac{p}{\lambda_n} = \frac{2\pi f}{2\pi f_n} = \frac{25}{30} = 0.833$, we have,

$$\begin{aligned} A_0 &= \frac{(50^{-6}\,\mathrm{N})/(90\,\mathrm{N/m})}{\sqrt{(2\cdot 0.32\cdot 0.833)^2 + (1-0.833^2)^2}} \\ &= 9.04 \times 10^{-7}\,\mathrm{m} = 0.904\mu\mathrm{m}. \end{aligned}$$

Similarly, we find the phase as,

$$\begin{aligned} \phi &= \tan^{-1}\frac{2\cdot 0.32\cdot 0.833}{1-0.833^2} \\ &= 1.05\,\mathrm{rad}. \end{aligned}$$

$m = 2.533 \times 10^{-9}$ kg, $\xi = 0.32$, $A_0 = 0.904\mu$m, and $\phi = 1.05$ rad