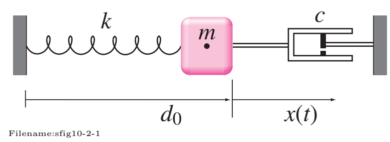

Figure 10.10: Spring-mass dashpot.

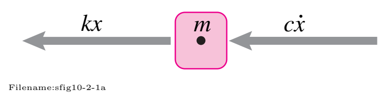

Figure 10.11: Free body diagram of the mass at an instant t.

SAMPLE 10.3 A block of mass 10 kg is attached to a spring and a dashpot as shown in Figure 10.11. The spring constant $k = 1000\,\text{N/m}$ and a damping rate $c = 50\,\text{N·s/m}$. When the block is at a distance d_0 from the left wall the spring is relaxed. The block is pulled to the right by 0.5 m and released. Assuming no initial velocity, find

1. the equation of motion of the block.
2. the position of the block at $t = 2\,\text{s}$.

Solution

1. Let x be the position of the block, measured positive to the right of the static equilibrium position, at some time t. Let $\dot{x}$ be the corresponding speed. The free body diagram of the block at the instant t is shown in Figure 10.12.

   Since the motion is only horizontal, we can write the linear momentum balance in the x-direction ($\sum F_x = m\,a_x$):

$$\underbrace{-kx - c\dot{x}}_{\sum F_x} = m \underbrace{\ddot{x}}_{a_x}$$

$$\text{or} \quad \ddot{x} + \frac{c}{m}\dot{x} + \frac{k}{m}x = 0 \qquad (10.16)$$

   which is the desired equation of motion of the block.

$$\ddot{x} + \tfrac{c}{m}\dot{x} + \tfrac{k}{m}x = 0.$$

2. To find the position and velocity of the block at any time t we need to solve Eqn (10.16). Since the solution depends on the relative values of m, k, and c, we first compute c^2 and compare with the *critical value* $4mk$.

$$\begin{aligned} c^2 &= 2500(\text{N·s/m})^2 \\ \text{and} \quad 4mk &= 4{\cdot}10\,\text{kg}{\cdot}1000\,\text{N/m} = 4000(\text{N·s/m})^2. \\ \Rightarrow \quad c^2 &< 4mk. \end{aligned}$$

   Therefore, the system is underdamped and we may write the general solution as (see box 10.2 on page 501)

$$x(t) = e^{-\frac{c}{2m}t}\,[A\cos\lambda_D t + B\sin\lambda_D t] \qquad (10.17)$$

   where

$$\lambda_D = \sqrt{\frac{k}{m} - \left(\frac{c}{2m}\right)^2} = 9.682\,\text{rad/s}.$$

   Substituting the initial conditions $x(0) = 0.5\,\text{m}$ and $\dot{x}(0) = 0\,\text{m/s}$ in Eqn (10.17) (we need to differentiate Eqn (10.17) first to substitute $\dot{x}(0)$), we get

$$\begin{aligned} x(0) &= 0.5\,\text{m} = A. \\ \dot{x}(0) &= 0 = -\frac{c}{2m}\cdot A + \lambda_D{\cdot}B \\ \Rightarrow \quad B &= \frac{A\,c}{2m\lambda_D} = \frac{(0.5\,\text{m}){\cdot}(50\,\text{N·s/m})}{2{\cdot}(10\,\text{kg}){\cdot}(9.682\,\text{rad/s})} = 0.13\,\text{m}. \end{aligned}$$

   Thus, the solution is

$$x(t) = e^{(-2.5\frac{1}{\text{s}})t}\,[0.50\cos(9.68\,\text{rad/s}\;t) + 0.13\sin(9.68\,\text{rad/s}\;t)]\ \text{m}.$$

   Substituting $t = 2\,\text{s}$ in the above expression we get $x(2\,\text{s}) = 0.003\,\text{m}$.

$$x(2\,\text{s}) = 0.003\,\text{m}.$$